On polynomials interpolating between the stationary state of a $O(n)$ model and a Q.H.E. ground state

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Abstract

We obtain a family of polynomials defined by vanishing conditions and associated to tangles. We study more specifically the case where they are related to a $O(n)$ loop model. We conjecture that their specializations at $z_i = 1$ are *positive* in n. At $n = 1$, they coincide with the the Razumov-Stroganov integers counting alternating sign matrices.

We derive the CFT modular invariant partition functions labelled by Coxeter-Dynkin diagrams using the representation theory of the affine Hecke algebras.

1 Introduction

Much progress has recently been made in the study of the ground state of the XXZ spin chain Hamiltonian when the anisotropy parameter is equal to $1/2$ [1]. The Hamiltonian is closely related to a stochastic Hamiltonian of a $O(1)$ fully packed loop model [2, 3].

The components of the $O(1)$ model Hamiltonian stationary state can be normalized to be positive integers, and it is conjectured (the Razumov-Stroganov (R-S) conjecture [4]) that these integers are in bijective correspondence with the states of a square-ice-model with domain wall boundary conditions, or equivalently with certain classes of alternating-sign matrices.

In previous works [6][7], we have introduced the polynomials discussed in this paper. In [6], they were discovered through the study of some representations of the double-affine Hecke algebra [8]. In [7], they were obtained by deforming the $O(1)$ transfer matrix eigenstates of Di Francesco and Zinn-Justin [5] in order to generalize the the R-S conjecture to the $O(n)$ case.

The aim of this paper is to further study these polynomials:

- We construct a family of polynomials which transform linearly under the braid group. We single out a basis in correspondence with the flat tangles or patterns.
- We define a deformation of the R-S integers by evaluating the basis polynomials at 1. We observe they enjoy positivity properties suggesting that they may coincide with a weighted enumeration sum of objects related to alternating sign matrices.

The polynomials can be defined by the vanishing conditions which they obey when several variables come close to each other. These vanishing conditions, called the wheel conditions [10], have been classified in [6] in relation with the representation theory of affine Hecke algebras. The number of variables involved in the wheel condition is $k+1$ and the wheel condition depends on another parameter r related to the degree with which the polynomial vanishes when the points are put together. In an orthogonal basis, the polynomials are non-symmetric Macdonald polynomials at specialized values of their parameters $t^{k+1}q^{r-1}=1$.[1]

Another way to obtain them, giving rise to a different basis, is by deforming the $O(1)$ model transfer matrix ground state. When $k=2$, the polynomial representation is dual to a representation of the Temperley and Lieb (T.L.) algebra of a loop model (defined on the disc) parameterized by $n=-2\cos(\theta)$ where n is the fugacity of the loops. The polynomials basis we consider here is dual to the loop model basis. Its specialization at $n=1$ coincides with the components of the $O(1)$ model transfer matrix ground state.

The basis polynomials are obtained from the action of difference operators [17] on a generating state which in the $r=2$ case is simply a product of q-deformed Vandermonde determinants. They can also be defined as the Kazhdan-Lusztig (K-L) basis [15][16] of this representation.[2]

At $\theta=0$, the wheel conditions are precisely the constraint imposed by the interactions on the Q.H.E. wave functions and the generating polynomial of the basis coincides with a wave functions of the Quantum Hall Effect (Q.H.E) [13]. In this context, the variables are the coordinates of particles distributed in k layers (or spins) not interacting with each other.

We generalize the construction to the case of a cylinder. The T.L. representations acting on link patterns depend on a second parameter n' equal to the weight assigned to the loops which wind around the cylinder. We study the simplest case where $n'=2\cos(\theta/2)$,

[1] The non-symmetrical Macdonald polynomials were first introduced in the context of the Calogero-Sutherland models by diagonalizing the affine Hecke generators $\bar{y}_i$ (constructed in appendix C) in the published version of [9]. They have been extensively used by Cherednick [8].

[2] The relation with the Kazhdan-Lusztig basis has been explained to us by A. Lascoux.

and in the case where $n = n' = 1$, we recover the stationary state of a $O(1)$ model considered by Mitra and Nienhuis [20, 21].

The specialization of these polynomials when all their variables are set equal to one turns out to be polynomials in n in the disc case and n' in the cylinder case, with integer coefficients. We observe on examples and conjecture in general that these integers are *positive*. When $n = n' = 1$, the disc and the cylinder polynomials become respectively the integers previously conjectured to count alternating sign matrices and half-turn alternating sign matrices in different topological sectors [4][22].

In the simple case where the Hecke algebra representation is generated from the action of difference operators on a product of q-Vandermonde determinants, we have verified that the evaluation at $z_i = 1$ of the K-L basis polynomials are positive.

Although not directly related to these polynomials, we derive the $c < 1$ unitary modular invariant partition functions of conformal field theories by decomposing certain representations of the affine Temperley and Lieb algebra acting on Dynkin diagrams into irreducible representations. As a byproduct, we define an action of the modular group on the irreducible representations of the affine T.L. (and more generally Hecke) algebra which relates them to a tensor product of Virasoro representations.

The paper is organized as follows.

The first part is introductory and serves as a motivation, we describe the representations of the affine T.L. algebra acting on a system of lines projected onto the disk. We study in more detail the case of the punctured disc.

The second and third part are the core of the paper and can to a large extend be read independently. In the second part, we obtain representations dual to those of the first part in terms of polynomials obeying the wheel conditions. We give examples where these polynomials are deformed Q.H.E. wave functions. In the third part, we state the positivity conjectures when the variables of the polynomials are set equal to one.

In the fourth part, we consider the case $n = 1$, and we show that the polynomials are the components of a $O(1)$ loop model ground state. In the fifth part, we introduce the representations of the affine T.L. algebra on SOS paths which diagonalize the affine generators. In the sixth part, we consider more specifically the unitary representation when the parameter of the algebra is a root of unity. We decompose certain representations acting on Dynkin diagram paths into irreducible representations.

2 Representation of the affine T.L. algebra on patterns

The basic definitions of an affine Hecke algebra (A.H.A.) $\mathcal{A}_N(t)$, and its T.L. restriction $\mathcal{A}_N^T(t)$, where t is a parameter, are given in the appendix A. In this section, we define tangles and patterns which are the natural objects to represent the action of the generators of $\mathcal{A}_N^T$. The tangles carry the three dimensional topological information. They can be decomposed into a linear combination of planar tangles called patterns. The patterns provide a basis of a representation of the affine T.L algebra which we describe in this section.

2.0.1 Tangles

A tangle is made by a set of open strings embedded into a three dimensional ball such that their extremities are on the boundary. We also consider the case where the ball is pierced by a flux running through it. A string can eventually connect the center of the ball or the flux to the boundary. The strings cannot cross each other, they cannot cross the flux either. Two tangles are considered equivalent if there is an ambient isotopy of one tangle to the other keeping the boundary of the 3-ball (and the flux) fixed.

The extremities of the strings on the boundary are arranged into N marked points $1, 2, \ldots, N$ ordered anticlockwise around the great circle. They are represented by their projection onto the flat disc bounded by the great circle. By noting the under and over crossings, one defines a tangle diagram. To respect isotopy invariance, tangle diagrams are also identified through equivalence relations known as Reidemeister moves. These moves are represented on figure (1) (Strictly speaking, the first Reidemeister move does not involve the factor $-t^{-\frac{3}{4}}$).

In the punctured case we project the flux onto the origin and therefore, the projected strings are not allowed to cross the origin.

2.0.2 Linear operators

Linear operators can be represented by an annulus with N cyclically ordered marked points $(\bar{1}, \ldots, \bar{N})$ on its inner boundary and M cyclically ordered marked points $(1, \ldots, M)$ on its outer boundary connected by strings (a string can connect two points of the same boundary). The action of the annulus on the disk with N marked points is null if $M \neq N$. It is obtained by gluing its inner boundary to the boundary of the disk so as to identify the marked points with those of the disk: $(1, \ldots, N) = (\bar{1}, \ldots, \bar{N})$, and by joining the

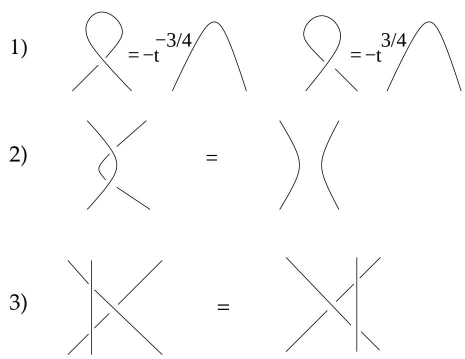

Figure 1: Reidemeister moves corresponding to the equations (2),(3).

strings of the disk with the strings of the annulus ending at the same point. For example, the identity is represented by disjoined open strings connecting i to $\bar{i}$.

The braid group generators:

$$\begin{aligned} g_i &= t^{-\frac{1}{4}} T_i = t^{\frac{1}{4}} + t^{-\frac{1}{4}} e_i, \\ g_i^{-1} &= t^{\frac{1}{4}} T_i = t^{-\frac{1}{4}} + t^{\frac{1}{4}} e_i, \end{aligned} \tag{1}$$

are represented by an annulus with $N-2$ disjoined strings connecting $\bar{l}$ to l for $l \neq k, k+1$, a string connecting $\overline{k+1}$ to k, and a string connecting $\bar{k}$ to $k+1$ passing over it (see figure 3).

The two first Reidemeister moves figure (1) result from the relations:

$$\begin{aligned} e_i g_i &= -t^{-\frac{3}{4}} e_i \\ g_i g_{i\pm 1} e_i &= e_{i\pm 1} e_i \quad , \end{aligned} \tag{2}$$

and the third move is the braid relation:

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i \ \text{ if } |i-j| > 1. \end{aligned} \tag{3}$$

A map from operators acting on the disc with N marked points to operators acting on the disc with $N+1$ marked points consists in adding an additional string to the pattern connecting $\overline{N+1}$ to $N+1$ without adding any crossing.

Conversely, a partial trace is defined by joining together the extremities N and $\bar{N}$ without adding any crossings. It maps operators acting on the disc with N marked points to operators acting on the disc with $N-1$ marked points.

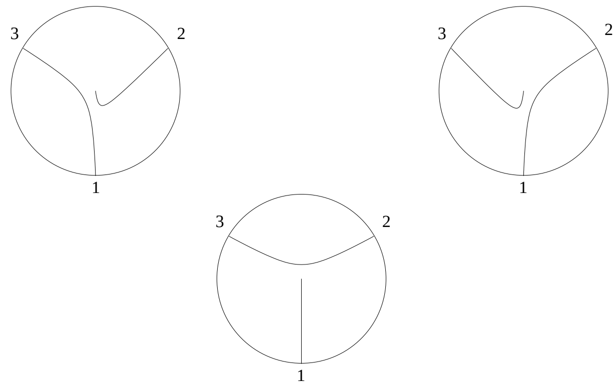

Figure 2: The patterns $\beta\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\beta\alpha$.

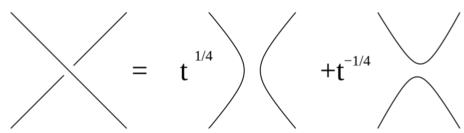

Figure 3: Skein relation corresponding to equation (1). The figure obtained by rotating each piece by 90 degree is also valid.

2.0.3 Patterns

A pattern is tangle with a flat projection on the disc. It can be represented by disjoint lines connecting pairwise the boundary points of the disk. Also, a line can start vertically down from the inside to reach the boundary without winding (figure 2).

We can encode a pattern π by a string of letters α or β [16][20]. We put α for i if the marked point is connected to the inside. In the disc case, we put α for i and β for j when $i < j$ are connected by a line. Therefore, when we cut the string into two pieces, the left piece must contain at least as many α's as β's. In the the punctured disc case, the lines are oriented so that the domain bounded by it and the great circle surrounds the puncture anticlockwise and we put α for the beginning and β for the end of the line.

Given a string of α's and β's, by successively erasing factors $\alpha\beta$, one can recover the position of the isolated α's.

It will often be convenient to view a pattern as an infinite periodic string with the identification $\pi_{i+N} = \pi_i$.

We denote $\mathcal{H}_N$, the vector space made by linear combinations of these patterns.

Tangles can be projected onto patterns as follows: Using the skein relation of figure (3), one can represent a tangle with at least one crossing as a linear combination of two tangles with one less crossing. A loop not surrounding the origin and not crossing any other loop is removed by multiplying the weight of a pattern by $\tau = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$.

In the punctured disc case, if N is even a loop surrounding the origin and not crossing any other loop is removed by multiplying it by $u+u^{-1}$, where u is a new parameter. If N is odd, we require that if one rotates by an angle 2π around the origin a line which starts from it, the weight is multiplied by u. These transformations preserve the equivalence under Reidemeister move, and using them, a tangle can be projected onto a linear combinations of patterns.

We define linear operators acting in $\mathcal{A}_N^T$ as for tangles. In particular, the T.L. generators e_k are represented by an annulus with $N-2$ disjoined lines connecting $\bar{l}$ to l for $l \neq k, k+1$, a line connecting $\bar{k}$ to $\overline{k+1}$, and a line connecting k to $k+1$. Using the above rules, it is straightforward to verify the T.L. relations with diagrams:

$$\begin{aligned} e_i^2 &= \tau e_i, \\ e_i e_j &= e_j e_i \text{ if } |i-j| \geq 2, \\ e_i e_{i\pm 1} e_i &= e_i. \end{aligned} \tag{4}$$

2.0.4 Temperley Lieb representations

We give the explicit expression of the T.L. matrices e_i in the link pattern basis.

We view a pattern as a string of N letters of α and β. If the difference between the number of α's and β's is larger than or equal to two, it is not conserved under the action of the T.L. algebra. Here, we restrict to the case where this difference is equal to zero or one (see figure 4 for different representations of the patterns).

Let us first consider the disc case. Following Lascoux and Schützenberger [16], we identify the link patterns basis with a Kazhdan-Lusztig (K-L) basis (see section 3.5.1).

A pattern π carries the label i if it is an eigenstate of e_i, in other words if $\pi_i\pi_{i+1} = \alpha\beta$. Given a pair of patterns (π, π'), we associate to it a reduced pair (π^r, π'^r), by successively erasing the factors $\alpha\beta$ located at the same place of the two strings. Two patterns are said to be matched if their reduced expression differ only by a change of a factor $\alpha\beta \to \beta\alpha$. It can be encoded into a K-L graph [15], having the patterns as vertices and an edge between two vertices when they are matched (see figure 5). We define $\mu(\pi, \pi') = 1$ or 0 indicating if π and π' are matched or not.

The expression of the T.L. matrices e_i is given by:

$$e_i\pi \;=\; \tau\pi, \quad \text{if } \pi \text{ has label } i,$$

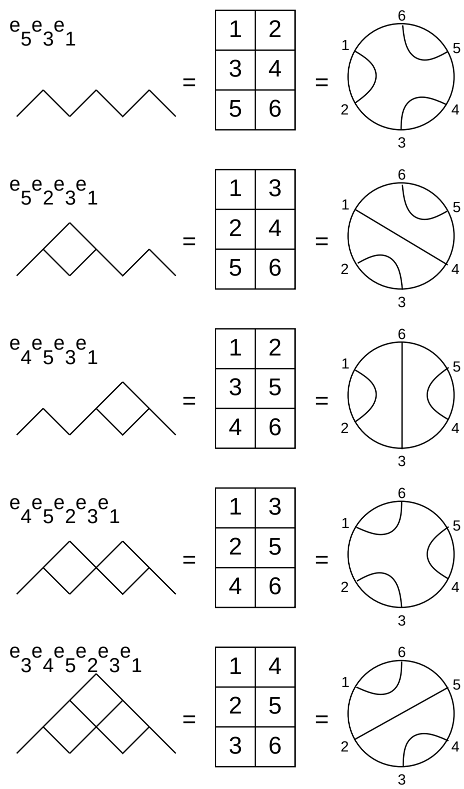

Figure 4: Several representations of the Hilbert space $\mathcal{H}_6$ describing six particles on the disc, in terms of:
a) T.L. words or paths.
b) Young Tableaux where the two columns give the positions of the α's and β's of the string.
c) Patterns.

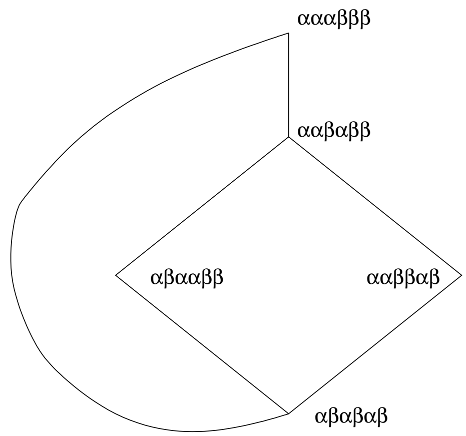

Figure 5: Kazhdan-Lusztig graph for 6 particles on the disc ($\mathcal{H}_6$).

$$= \sum_{\nu \text{ has label } i} \mu(\pi,\nu)\nu, \text{ if } \pi \text{ has not label } i. \tag{5}$$

In the punctured disc case, this construction is modified as follows. We view a pattern as an infinite periodic string of letters α, β. The reduction proceeds as in the disc case where we take into account the periodicity of the patterns: $\pi_{i+N} = \pi_i$. One still has the expression (5) of the T.L. matrices, where $\mu(\pi,\pi') = 0$ if the patterns are not matched and the value of $\mu(\pi,\pi')$ for matched pattern is given by the following rules.

In the N even case, $\mu(\pi,\pi') = 1$, except if the two reduced strings are $\alpha\beta$ and $\beta\alpha$ where $\mu(\pi,\pi') = -(u+u^{-1})$.

In the N odd case, $\mu(\pi,\pi') = 1$, except if the two reduced strings are $\pi^r = \beta\alpha\alpha$ and $\pi'^r = \alpha\beta\alpha$ where $\mu(\pi,\pi') = u$, $\pi^r = \beta\alpha\alpha$ and $\pi'^r = \alpha\alpha\beta$ where $\mu(\pi,\pi') = u^{-1}$, and the permuted cases where $\mu(\pi,\pi') = \mu(\pi',\pi)^{-1}$.

In the appendix B, we give the matrices representing $\mathcal{A}^T_{2,3}$ which follow from the above rules.

2.0.5 Affine generators in the pattern representation

To obtain the affine algebra representation [29], let us define the cyclic operator σ which rotates the ball by $-2\pi/N$ around its axis. It acts on patterns by shifting the extremities of the strings by one unit clockwise: $i \to i-1$, so that we have:

$$\sigma g_i = g_{i-1}\sigma, \tag{6}$$

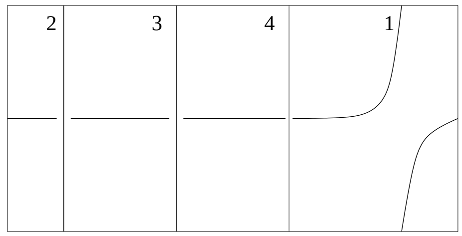

Figure 6: Affine generator y_1.

for $i \geq 2$, and one can define a new braid generator $g_N = \sigma g_1 \sigma^{-1}$. In the punctured disc case, σ acts by shifting the indices $i \to i-1$.

Let us construct the affine generators y_i which form a family of commuting operators, and are therefore useful to characterize the states of a representation. They also have a topological interpretation.

We define y_1 as:

$$y_1 = T_1 T_2 \dots T_{N-1} \sigma. \tag{7}$$

Let us define $y'_1 = t^{-\frac{N-1}{4}} y_1 = \sigma g_2 g_3 \dots g'_N$. It acts on tangles by letting the extremity of the string ending at the marked point 1 wind by an angle -2π around the boundary of the disk underneath the strings ending at position $j \neq 1$ (see figure 6). Similarly, we define y_i from the relations defining the affine Hecke algebra (80) of appendix A, and $t^{\frac{2j-N-1}{4}} y_j$ lets the extremity of the string ending at the marked point j rotate by an angle -2π around the boundary of the disk above the strings ending at $j < i$ and underneath the strings ending at $j > i$.

Let us consider the action of y_1 in the pattern representation.

If N is even, the line connected to the marked point 1 remains underneath all the other lines and we can decouple it from them. It is therefore sufficient to consider a pattern with only this line. It is straightforward to verify that $y'_1 = -t^{-\frac{3}{4}}$ if there is no puncture. In the punctured case, the basis has two elements, $\beta\alpha$ and $\alpha\beta$, where the line surrounds the puncture in two different ways and the matrix has the expression (see appendix B):

$$y'_1 = \begin{pmatrix} 0 & t^{\frac{1}{4}} \\ -t^{-\frac{3}{4}} & (u+u^{-1})t^{-\frac{1}{4}} \end{pmatrix}. \tag{8}$$

It therefore satisfies the quadratic relation:

$$(y'_1 - u^{-1} t^{-\frac{1}{4}})(y'_1 - u t^{-\frac{1}{4}}) = 0. \tag{9}$$

We define a second parameter τ' equal to the weight of a loop surrounding the puncture. It is obtained by sandwiching y'_1 (8) between two e_1:

$$e_1 y'_1 e_1 \;\; = \;\; -\tau' t^{-\frac{3}{4}} e_1. \tag{10}$$

Therefore, in the N even case, we have:

$$\tau' = u + u^{-1}. \tag{11}$$

In the case where N is odd, we can proceed similarly by considering a basis with three elements involving the patterns of figure 2. The matrix representing y_1 is given in the appendix B. The relations (9) and (11) are modified and become:

$$(y'_1 - u^{-1}t^{-\frac{1}{2}})(y'_1 - u) = 0. \tag{12}$$

$$\tau' = ut^{\frac{1}{4}} + u^{-1}t^{-\frac{1}{4}}. \tag{13}$$

In both cases, $\tau' = x + x^{-1}$, where $x^2 = y_+/y_-$ is the ratio of the two eigenvalues of y_1.

2.0.6 Involutions

One can define two natural involutions $\mathcal{F}$ and $\mathcal{T}$ on tangles.

The flip $\mathcal{F}$ rotates the ball by half a turn around a horizontal axis passing through its center and relabels the points $k \to N+1-k$. It induces the isomorphism taking: $g_k \to g_{N-k}$ and $\sigma \to \sigma^{-1}$.

The reflection $\mathcal{T}$ reflects the ball through its equator. It induces the antilinear involution taking: $g_k \to g_k^{-1}$, $t \to t^{-1}$, and σ to itself.

2.0.7 Inclusion and Conditional expectation value

There is an imbedding of $\mathcal{H}_{N-2}$ into $\mathcal{H}_N$ which takes x to $e_1 x$. On patterns, it consists in shifting by two the labels of a pattern of $\mathcal{H}_{N-2}$, and by adding a line connecting two new points labelled 1 and 2 in such a way that $1, 2, \ldots, N$ are cyclically ordered. Thus, it adds the two letters $\alpha\beta$ at the beginning of the string representing the pattern.

Conversely, there is a projection E called conditional expectation value [28] from $\mathcal{H}_N$ to $\mathcal{H}_{N-2}$ defined by:

$$e_1 \pi = \tau E(\pi) e_1. \tag{14}$$

E connects the lines ending at $1, 2$, so as to produce a pattern starting with $\alpha\beta$, which is identified with a pattern in $\mathcal{H}_{N-2}$ by the preceding imbedding. This projection is hermitian for the scalar product previously defined and commutes with the action of $\mathcal{A}_{N-2}^T$.

3 Polynomial representations

We consider here polynomials in as many variables as there are line extremities attached to the boundary of a pattern. One of our aims is to identify representations of $\mathcal{A}_N^T$, dual to those of the preceding sections, acting on these polynomials. This is done through the introduction of a vector Ψ whose components are polynomials indexed by the patterns, and by requiring that Ψ transforms in the same way under the two actions of the generators, on patterns or on polynomials. To obtain these irreducible representations of $\mathcal{A}_N$, we restrict the polynomials to obey some conditions called the wheel conditions.

3.1 The vector Ψ.

The main problem of this section is to obtain a vector Ψ:

$$\Psi = \sum_{\pi} \pi \psi_\pi(z_i), \tag{15}$$

constructed in the following way. The vectors π are basis vectors of a representation on which $\mathcal{A}_N$ acts from the left. The ψ_π are polynomials on which it acts from the right. We denote with a bar the right action of $\mathcal{A}_N$ on polynomials to distinguish it from the left action. We want to determine the ψ_π in such a way that both actions give the same result on the vector Ψ:

$$\begin{aligned} \Psi\bar{e}_i &= e_i\Psi \\ \Psi\bar{\sigma} &= \sigma\Psi, \end{aligned} \tag{16}$$

Said differently, we identify the polynomials with the dual basis $\psi_\pi(z_i)$ of the representation. These conditions are equivalent to the conditions (53) introduced later.

Using a physical picture, $\psi_\pi(z_i)$ is the amplitude to find the particles labelled by i at positions z_i in the Resonance valance bond state (R.V.B.) π where the lines represent the spin singlets. The first condition (16) is the q-deformed conditions for Ψ to be a bosonic wave function.

For the disc and the punctured disc representations of section 2.0.3, this problem is solved in section 3.5.

3.2 The polynomials.

We consider here Laurent polynomials in N variables $z_i^{\pm 1}$, $1 \leq i \leq N$, which we identify with the generators z_i of the double A.H.A. of appendix A. We exhibit a representation of the double A.H.A. on these polynomials depending on two parameters t and q.

We define permutation operators which permute the labels of the variables z_i:

$$\begin{aligned} z_i s_i &= s_i z_{i+1}, \\ z_{i+1} s_i &= s_i z_i, \text{ and} \\ z_l s_i &= s_i z_l \text{ if } l \neq i, i+1, \end{aligned} \tag{17}$$

Using the construction of appendix A.1.1, one recovers the Lascoux and Schützenberger [17] expression of Hecke generators $\bar{e}_i$ (for $1 \leq i \leq N-1$) acting from the right of an expression as follows:

$$\begin{aligned} \bar{e}_i &= (t^{\frac{1}{2}} z_{i+1} - t^{-\frac{1}{2}} z_i)(1 - s_i)(z_i - z_{i+1})^{-1} \\ \bar{e}_i - \tau &= (1 - s_i)(z_i - z_{i+1})^{-1}(t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1}), \end{aligned} \tag{18}$$

Therefore, $\bar{e}_i$ projects onto polynomials symmetrical under the exchange of z_i and z_{i+1}, and $\tau - \bar{e}_i$ onto polynomials divisible by $(t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1})$. Notice that the generators $\bar{e}_i$ obey the Hecke, not the T.L. relations.

It is convenient to define variables $z_i,\ i \in \mathbb{Z}$, which are cyclically identified as:

$$z_{i+N} = q^{-1} z_i. \tag{19}$$

We can represent the cyclic operator $\bar{\sigma}$ as:

$$z_i \bar{\sigma} = \bar{\sigma} z_{i+1}, \tag{20}$$

and let it act as the identity when it is located left of an expression:

$$1\bar{\sigma} = c_0. \tag{21}$$

Thus, on a homogeneous polynomials of total degree $|\lambda|$, $\bar{\sigma}^N = c_0^N q^{-|\lambda|}$, where c_0 needs to be adjusted to satisfy the duality condition (16).

In the appendix (C) we reproduce the results of [9] showing that for the dominant ordering of the monomials, the operators $\bar{y}_i$ are represented by triangular matrices and we obtain their spectrum for t, q generic. We summarize the main results here:

- A partial ordering on monomials can be defined as follows [12].

  Given a monomial $z^\lambda = z_1^{\lambda_1} \dots z_N^{\lambda_N}$, we consider the partition λ^+ obtained by re-ordering the λ_i in decreasing order.

  We say that $\lambda \geq \mu$ if the following conditions are satisfied.

  The partition λ^+ is larger than μ^+ for the dominance order defined as:

$$\lambda_1^+ + \dots \lambda_i^+ \geq \mu_1^+ + \dots \mu_i^+ \text{ for all } i \geq 1. \tag{22}$$

  When $\lambda^+ = \mu^+$, μ can be obtained from λ through a sequence of transformations $(\lambda'_i, \lambda'_{i+1}) \to (\lambda'_{i+1}, \lambda'_i)$ with $\lambda'_i > \lambda'_{i+1}$.

- The polynomial representations of the A.H.A. depend on the two parameters, t and q, and on a partition λ^+.

  The affine generators y_i are lower triangular for the preceding order.

  An orthogonal basis is obtained by diagonalizing the y_i simultaneously and a basis state $F_\pi(z_i)$ is characterized by its highest degree monomial:

$$F_\pi(z_i) \propto z_1^{\lambda^+_{\pi_1}} \dots z_N^{\lambda^+_{\pi_N}} + \dots, \tag{23}$$

  where π is a permutation of λ^+: $(\lambda_\pi)_i = \lambda^+_{\pi_i}$.

- Up to an overall normalization factor, the eigenvalues of the operators y_k on the polynomials F_π are obtained by permuting the eigenvalues $\hat{y}_a$ of y_a on the highest weight polynomial F_1 :

$$\begin{aligned} F_\pi y_k &= F_\pi \hat{y}_{\pi_k}, \text{ with :} \\ \hat{y}_a &= q^{-\lambda_a^+} t^{(a-1)}. \end{aligned} \tag{24}$$

3.3 Wheel condition

In this section, we introduce the vanishing conditions obeyed by the polynomials which enable us to construct the vector Ψ. These vanishing conditions, called the wheel conditions are studied in [6]. When certain conditions are obeyed by the parameters t, q, the space of polynomial obeying these wheel conditions form a representation of the A.H.A.. We give their definition and we explain why they are preserved under the action of the A.H.A.. We motivate the vanishing conditions from the Q.H.E. point of view by studying some examples in the next section.

3.3.1 Definition of the wheel conditions

Fix two integers, k and r, and two variables t and q related by:

$$t^{k+1}q^{r-1} = 1. \tag{25}$$

If m is the largest common divisor of k+1 and $r-1$, we take $t^{\frac{k+1}{m}}q^{\frac{r-1}{m}} = \omega$, with ω a primitive m^{th} root of unity.

We say that:

A Laurent polynomial $P(z_i)$, in N variables $z_i^{\pm 1}$ satisfies the wheel condition (k, r), if:

- For any subset of $k+1$ indices $\{i_a\},\ 1 \le a \le k+1$.
- For any set of k integers $b_{aa+1} \in \mathbb{N},\ 1 \le a \le k$, such that:

$$\begin{aligned} a) &\quad b_{aa+1} = 0 \ \Rightarrow\ i_{a+1} > i_a, \\ b) &\quad \sum_1^k b_{aa+1} \le r-2. \end{aligned} \tag{26}$$

- $P(z_1, \dots z_N) = 0$ when we restrict the variables to satisfy the wheel conditions:

$$z_{i_{a+1}} = tq^{b_{a,a+1}} z_{i_a}. \tag{27}$$

A set $\{i_a\}$,$\{b_{aa+1}\}$ satisfying the conditions (26) defines an admissible wheel, and the vanishing condition specified by this wheel is called a wheel condition.

We shall mostly be interested in the simplest cases $r = 2$ where the rule simplifies drastically. One has $q = t^{-(k+1)}$ and given k+1 ordered indices, $1 \le i_1 \le i_2 \dots \le i_{k+1} \le N$, the polynomial must vanish when $(z_{i_1}, z_{i_2}, \dots, z_{i_{k+1}}) = (z, tz, t^2z, \dots, t^kz)$.

In the appendix D we show that the wheel conditions are preserved under the action of the A.H.A.

3.3.2 Admissibility conditions

Here, we recall the results of [6] about the polynomial representations of the A.H.A. when the wheel conditions are satisfied.

When the condition (25), $t^{k+1}q^{r-1} = 1$, is satisfied, and for t generic, the representation admits an irreducible subrepresentation on polynomials satisfying the wheel condition (3.3). The basis states are eigenstates of the affine generators y_i (98) and are proportional to the non-symmetrical Macdonald polynomials [9][25] specialized at $t^{k+1}q^{r-1} = 1$. They are characterized by their highest weight monomial now subject to more restrictive admissibility conditions [6]:

- A partition $\lambda^+ = (\lambda_1^+, \ldots, \lambda_N^+)$ defines an admissible state if it satisfies:

$$\lambda_a^+ - \lambda_{a+k}^+ \geq r-1, \ \forall \ a \leq N-k. \tag{28}$$

- A highest weight monomial is characterized by the shortest admissible permutation π such that one has: $(\lambda_\pi)_i = \lambda_{\pi_i}^+$. A permutation π defining λ_π is admissible if it satisfies the condition:

$$\lambda_a^+ - \lambda_{a+k}^+ = r-1 \Rightarrow a = \pi_i, \ a+k = \pi_j \text{ with } j > i. \tag{29}$$

  In other words, the weight $\lambda_\pi = (\lambda_{\pi_1}^+, \ldots, \lambda_{\pi_N}^+)$ is a permutation of λ^+ such that λ_a^+ remains to the left of λ_{a+k}^+ whenever $\lambda_a^+ - \lambda_{a+k}^+ = r-1$.

In the the next section, we describe how the states of this representation can be encoded into tableaux, and in section 6.1, we shall give a path description valid the case of the two columns tableaux.

3.3.3 Tableaux

The tableaux [38][39] give a convenient way to represent the states of the A.H.A. representations considered in the last section.

An admissible permutation π determining the polynomial F_π can be encoded by distributing the numbers $i, \ 1 \leq i \leq N$, into the boxes of a planar diagram. The boxes are labelled from 1 to N and the polynomial F_π is represented by a tableau putting i in the box π_i. We identify the tableau representing the polynomial F_π with the permutation π.

The position of the box occupied by the number i determines the partial degree in the variable z_i and the eigenvalue of the affine generator $\bar{y}_i$ on F_π. Denote (x_a, x'_a), the cartesian coordinates of the box labelled a. Their sum labels the eigenvalue $\hat{y}_a$ of y_a (24), on the highest-weight polynomial F_0:

$$t^{x_a + x'_a} = \hat{y}_a, \tag{30}$$

The product of the second coordinate with $r-1$, $(r-1)x'_a$, is the degree of the highest-weight polynomial F_0 in the variable z_a. Thus:

$$\begin{aligned} x_a &= \lambda_a^+ \frac{k}{r-1} + a, \\ x'_a &= \frac{\lambda_a^+}{r-1}. \end{aligned} \tag{31}$$

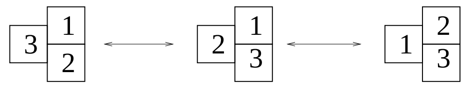

Figure 7: tableaux representing the states $(k, r) = (2, 3)$ obeying the extended wheel condition $p = 1$ for three particles. The left box is disconnected from the right boxes.

As explained in the appendix A.1.1, in order to obtain an irreducible representation, the rule of construction is such that a vertical move of one unit north or a horizontal move of one unit east has the effect to multiply the eigenvalue $\hat{y}_a$ by a factor t.

The value of x'_a modulo one, or equivalently the degree modulo $r - 1$, splits the boxes of a diagram into $r - 1$ classes, and it is convenient to split the diagram into $r - 1$ disconnected sub-diagrams.

The admissibility condition (29) can be rephrased into the rule:

- The numbers are strictly increasing down each column and across each row.

These are precisely the rules defining standard tableaux (see figure 4 and 7).

Starting from any polynomial, one generates the others by acting on it with intertwining operators Y_i described in A.1.1. The tableaux are transformed into: $F_\pi \to F_\pi Y_i = F_{\pi s_i}$ where s_i exchanges the positions of i and $i + 1$ in the tableau. Y_i acts only if the two boxes are not adjacent in the same row or column.

One can start from an arbitrary polynomial F_π of the basis to generate the other basis elements by acting with the Y_i upon it. In practise, the lowest degree polynomial has often a simple factorized expression and is a more convenient generator. In the next section, we exhibit some examples of these lowest degree polynomials.

3.4 Explicit solutions and Q.H.E. interpretation

We give some explicit solutions of the wheel conditions here. Although we use the Q.H.E. terminology to motivate them from a physical point of view, the polynomials of this section can easily be obtained independently of any connection with the Q.H.E.

We consider particles moving in the plane in a strong magnetic field projected to the lowest Landau level [35]. In a specific gauge the orbital wave functions are given by:

$$\psi_N(z) = \frac{z^k}{\sqrt{k!}} e^{-\frac{z\bar{z}}{4}}, \tag{32}$$

where $z = x + iy$ is the coordinate of the particle, and the magnetic length scale related to the strength of the magnetic field has been set equal to one. These orbitals are concentrated on shells of radius $\sqrt{k}$ occupying an area 2π. Each orbital is represented by a monomial z^k.

The quantum Hall effect [13] ground state Ψ is obtained by combining these individual orbitals into a many-body wave function. All the wave functions of system of N particles have a common factor $e^{-\sum_i \frac{z_i \bar{z}_i}{4}}$ which we omit. Thus, a monomial $z^\lambda = z_1^{\lambda_1} \dots z_N^{\lambda_N}$ describes a configuration where the particle j occupies the orbital λ_j. The wave functions are linear combinations of such monomials.

The physical properties are mainly characterized by the inverse filling factor which is the area occupied by a particle measured in units of 2π. It can only be defined in the thermodynamical limit, and is given by the limit when $N \to \infty$ of the maximum degree in each variable divided by the number of variables.

The effect of the interactions is to impose some vanishing conditions when the variables are in contact: $\Psi \sim (z_i - z_j)^m$ with m an integer when $z_i - z_j \to 0$. A ground state wave function is a polynomial of the minimal degree obeying the constraints. The difference between the polynomials considered here and the Q.H.E. are the wheel conditions which reduce to the Q.H.E. vanishing conditions when t and q tend to 1 (in which case, $k+1$ and $r-1$ must be prime numbers).

Through the examples considered here where the wave function has a product structure, it turns out that k has the interpretation of a spin index and $r-1$ has the interpretation of an inverse filling factor. The incompressibility condition translates into obtaining the representations of section 3.3.3 having the most compact diagrams.

- $k = 1$, solutions:

Let us denote $\Delta_t(z_1, \dots, z_k)$ the Vandermonde product:

$$\Delta_t(z_1, \dots, z_k) = \prod_{1 \le i < j \le k} (t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_j). \tag{33}$$

A solution obeying the r wheel condition has the simple product structure:

$$\psi_{k=1,r}(z_i) = \prod_{l=0}^{r-2} \Delta_{tq^l}(z_1, \dots, z_N). \tag{34}$$

It obviously satisfies the $k = 1$, r wheel conditions for wheels with $i_1 < i_2$, and if $i_2 < i_1$, the wheel condition follows from the fact that $(tq^{b_{12}})^{-1} = tq^{r-1-b_{12}}$.

This solution is the q-deformation of a Laughlin wave function with an inverse filling factor $r-1$ [13].

Let us pursue the Q.H.E. analogy further. If we insert a magnetic flux in the system at the origin, this has the effect to multiply this wave function by a factor $\prod_i z_i$. Thus, the orbital shells are expelled by one unit away from the origin and a region of area 2π is left vacant, which is as if $1/(r-1)$ particle had been removed from the origin. When the flux is inserted, the eigenvalue of the operator y_1 gets multiplied by a factor q^{-1}, which can be interpreted as the phase acquired by the wave function when the particle winds around the flux.

- $r=2$, solutions with k arbitrary:

Let us show in the simplest case $r=2$ that k has the interpretation of a layer (spin) index. Consider N particles (variables) split into k layers of x_l, $1 \leq l \leq k$ particles each. We denote z_{li}, $1 \leq i \leq N_l$ the coordinates of the particles in the layer l and we order the indices so that $li < l'j$ for $l < l'$. We say that two variables are in the same layer if they share the same index l. The particles of the same layer repel each other so that the wave function representing the system vanishes when the variables $z_{lj} = tz_{li}$ for $i < j$, and the particles belonging to different layers do not interact. A ground state wave function representing this system obeys the $(k, 2)$ wheel condition because given $k+1$ variables, two of them necessarily belong to the same layer.

A simple wave function obeying the vanishing conditions is thus given by:

$$\psi_{k,r=2}(z_i) = \prod_{l=1}^{k} \Delta_t(z_{l1}, \ldots, z_{lN_l}). \tag{35}$$

Let us consider the incompressible limit. We look for polynomials of the minimal degree obeying the wheel condition (the degree measures the extension of the wave function). Thus, we split the layers into two sets and set $k = k_1 + k_2$. We fill each of the k_1 first layers with $N-1$ particles and each of the last k_2 layers with N particles. According to the rules of the preceding section 3.3.3, the polynomials belong to the irreducible representation characterized by the skew-diagram, $k^N/1^{k_1}$, with k_1 columns of length $N-1$ and k_2 columns of length N. The polynomial (35) coincides with the lowest weight of this representation.

Notice that if we do not restrict to the minimal degree case and allow the number of particles to differ by more than one in between different layers, the representation is described by a tableau with more than k columns according to the rules of section 3.3.3. Unless we specify it, we shall not consider these cases here.

- $k = 2$, $r = 3$ solution:

The case $k = 2$ and r, represents a system of two layers with an inverse filling factor $r-1$, and for $r = 3$, is the q-deformation of the Haldane-Rezayi [14] wave function. We repeat the Q.H.E. construction in the q-deformed case here. There are $N = N_1 + N_2$ particles or variables split into 2 layers of N_1, N_2 particles each. We label the variables by li where $l = 1, 2$ is the layer index and i a particle index within each layer. We order them so that $1i < 2j$.

We split the wave function $\psi_{k=2,r=3}(z_i)$ into a product:

$$\psi_{k=2,r=3}(z_{1i}, z_{2j}) = \phi(z_{1i}, z_{2j})\Delta_t(z_{11}, \ldots, z_{1N_1})\Delta_t(z_{21}, \ldots, z_{2N_2}), \tag{36}$$

Due to the factors Δ_t, the wheel condition is satisfied if two indices i_a, i_{a+1} involved in the wheel belong to the same layer and $b_{aa+1} = 0$. In particular, this covers all the cases where the three variables belong to the same layer.

By inspection, the wheels left to be considered are those for which:

$b_{aa+1} = 0 \Rightarrow i_a$ is in the first layer and i_{a+1} is in the second layer.

Thus, if the two variables $1i$ and $2j$ belonging to different layers participate to a wheel, the ratio z_{2j}/z_{1i} is either t or $t^2q = (tq)^{-1}$. The last two are equal due to the condition $t^3q^2 = 1$ (25). Therefore, we must impose ϕ to vanish in these cases.

For ϕ, we consider an expression of the form:

$$\phi(z_{1i}, z_{2j}) = \prod_{i=1}^{N_1}\prod_{j=1}^{N_2} f(z_{1i}, z_{2j}) \prod_{k=1}^{\inf\{N_1,N_2\}} f^{-1}(z_{1i_k}, z_{2i_k}), \tag{37}$$

where $1i_k, 2i_k$ is a maximal pairing between particles of the the first layer with those of the second layer and $N_1 - N_2$ particles of the first (second) layer are not paired if $N_1 > N_2$ ($N_2 > N_1$). For f we take:

$$f(z, w) = (tz - w)(z - tqw). \tag{38}$$

In all the wheel cases considered above, two of the three particles participating to the wheel, say $1i$ and $2j$, belong to different layers and are unpaired. Since the wave function ϕ, contains the factor $f(z_{1i}, z_{2j})$ it vanishes for this wheel. Therefore, the wheel condition is always satisfied.

The degree of ϕ can be reduced by antisymmetrizing it over the variables of the same layer z_{li} and by dividing the result by $\prod_{l=1}^{2}\prod_{i<j}(z_{li} - z_{lj})$.

If we restrict to the minimal degree cases $N_1 = N_2 = N$ or $N_1 = N - 1, N_2 = N$, using the rules of the preceding section 3.3.3, we see that the polynomial obtained here

is the lowest weight state of the representations characterized by the (skew)-diagram, 2^N or $2^N/1$, with two columns of length N_1 and N_2.

When $N_1 = N_2$, the wave function is proportional the Gaudin Determinant [41]:

$$\psi_{k=2,r=3}^{N_1=N_2} = \mathrm{Det}\left(\frac{1}{(t^{\frac{1}{2}} z_{1i} - t^{-\frac{1}{2}} z_{2j})(t^{\frac{1}{4}} z_{1i} - t^{-\frac{1}{4}} z_{2j})}\right) \frac{\Delta_t(z_{11}, \ldots, z_{1r}) \Delta_t(z_{21}, \ldots, z_{2r})}{\prod_{i<j}(z_{1i} - z_{1j}) \prod_{i<j}(z_{2i} - z_{2j})}. \quad (39)$$

For $r > 3$, we were not able to obtain a simple trial polynomial obeying the $(2, r)$ wheel condition with the minimal degree. The polynomials obeying these wheel conditions can nevertheless be obtained by specializing the non-symmetric Macdonald polynomials at $t^{k+1} q^{r-1} = 1$.

3.4.1 Inserting a flux

The polynomials with $r > 2$ allow to describe tangles with a flux inserted.

Let us fix an integer p with $1 \leq p \leq r - 1$ and impose the additional wheel condition:

- The polynomials vanish at least as ϵ^p when two arbitrary variables z_i and z_j approach zero as ϵ.

This constraint is preserved under the action of the A.H.A.. It can be realized on the polynomial representations characterized by a diagram with two disjoined columns vertically shifted by $p/(r - 1)$ with respect to one-another.

This results from the fact that given a polynomial in this representation and a monomial in its expansion, the sum of any two of its degrees is greater or equal to p. This is true for the highest monomial of an eigenstate of the affine generators $\bar{y}_i$, and thus for any monomial in its expansion by the construction of the appendix C.1.

The $p = 1$ case can simply be obtained by multiplying the disc polynomial $\psi_{k=2}^{N_1,N_2}$ by the product $\prod_{i=1}^{N_l} z_{li}$ where $l \in \{1, 2\}$ is the column with the smallest length.

3.5 Link pattern basis

Here, we identify the polynomial basis dual to the pattern basis of section (2.0.3). In the disc case, the basis coincides with the K-L basis and in the punctured disc case, the construction needs to be modified.

3.5.1 Kazhdan-Lusztig basis

We first give the general construction of the K-L basis [15] which we then specialize to the case of the the disc patterns [16].

The generating polynomial is a product of q-Vandermonde determinants as in (35):

$$\psi_1(z_i) = \prod_{l=1}^{k} \Delta_t(z_{l1}, \ldots, z_{lN_l}), \tag{40}$$

and the N_l are sorted decreasingly. It is more convenient here to index the variables from 1 to $N = \sum_{l=1}^{k} N_l$ with $li \rightarrow N_1 + \ldots N_{l-1} + i$.

We construct a representation of the Hecke Algebra called Specht representation [19] by letting the generators $e_i - \tau$ (18) with $1 \leq i \leq N-1$ act on this polynomial. It has a K-L basis which we describe here.

The basis states are given in terms of standard tableaux associated to a Young diagram as in section 3.3.3. The lowest tableau, denoted 1, has its columns filled with consecutive numbers and corresponds to the polynomial ψ_1 (It corresponds to the lowest tableau of figure 4) . We number the boxes of diagram as in 1. The standard tableau π puts the number i in the box π_i. It is generated from the tableau 1 by a succession of elementary permutations $\pi \rightarrow \pi s_i$, where s_i exchanges the position of i and $i+1$, and acts only if $i+1$ is in a column to the right of i and not in the same line as i. This determines an order on tableaux, the dominance (or Bruhat) order: $\pi' \geq \pi$ if it can be written in the preceding way as $\pi' = \pi w$ with w a word in the s_i.

To the tableau π, we associate an element $\bar{T}_\pi$ in the Hecke algebra obtained by substituting the generator $\bar{T}_i$ to the permutation s_i in the above expression of π. We obtain a polynomial basis of the Specht module, labelled by tableaux, by letting $\bar{T}_\pi$ act on ψ_1:

$$F_\pi(z_i) = \psi_1 \bar{T}_\pi, \tag{41}$$

We define an antilinear (but preserving the order) involution by $\mathcal{T}'(\bar{T}_i) = \bar{T}_i^{-1}$, $t^{\frac{1}{2}} \rightarrow t^{-\frac{1}{2}}$ and $\mathcal{T}'(\psi_1) = \psi_1$. The K-L basis states ψ_π are obtained from the basis states F_π by a triangular transformation and are defined by the relations:

$$\begin{array}{rcl} \mathcal{T}'(\psi_\pi) & = & \psi_\pi \\ \psi_\pi - F_\pi & \in & \oplus_{\pi' < \pi} t^{-\frac{1}{2}} \mathbb{Z}[t^{-\frac{1}{2}}] F_{\pi'}. \end{array} \tag{42}$$

Let us make a few observation about the K-L basis.

The Specht representation coincides with the AHA representation of section 3.3.3 only in the minimal degree case where the length of the columns of the Young diagram differ by at most one. In this case the cyclic permutation $\bar{\sigma}$ (20) acts within the representation.

If we denote by J the subset of $\{1, \ldots N\}$ where we omit the numbers $\{N_1, N_1 + N_2, \ldots, N\}$, the polynomial (40) obeys:

$$\psi_1 \bar{T}_i = t^{\frac{1}{2}} \psi_1, \ \forall i \in J. \tag{43}$$

One can define a K-L basis for the induced module defined by the relation (43). The Specht representation is a subrepresentation of this induced module.

One can define a KL-graph as in section 3.5.1 and the expression (5) of the matrices e_i holds in the general case. This gives a practical way to construct the K-L basis. The tableau π carries the label i if i is in a column to the left of $i+1$. An incomplete K-L graph (The Young graph) is drawn by connecting π to π' when π and π' differ by the permutation of two consecutive numbers. The trial basis states are obtained recursively by acting with $\bar{e}_i - \tau$ on ψ_π carrying the label i and such that i and $i+1$ are not in the same line:

$$\psi_{\pi s_i} = \psi_\pi (\bar{e}_i - \tau) - \sum_{\nu < \pi \text{ without label } i} \mu(\pi, \nu) \psi_\nu. \tag{44}$$

Using the first relation (16), one constructs the matrices $e_i - \tau$ in the trial basis and one completes the KL-graph using (44). Due to the factor $t^{\frac{1}{2}}$ in (43), the basis may violate the second condition in (42) (equivalently, $e_i - \tau$ violates the non-negative integrality condition (5)). This is cured by completing the K-L graph with new links and by correcting the basis states (44) accordingly.

The dual basis is also a K-L basis. It is generated by induction from the highest tableau having its lines filled with consecutive numbers. In the dual construction, $t^{-\frac{1}{2}}$ must be substituted to $t^{\frac{1}{2}}$ in the relations (42) and (43) and the conjugate diagram must be substituted to the original diagram. So, the set J is replaced by $J' = \{1, \ldots N\} \setminus \{N'_1, N'_1 + N'_2, \ldots, N\}$ where the N'_i are the lengths of the lines.

Specializing to the two column diagrams, we identify the tableaux with the link patterns of section 2.0.3. A tableau is the link patterns where the numbers in the left column encode the position of the α's, and the numbers in the right column the positions of the β's as in figure 4. The lowest tableau is the pattern having all the α's precede the β's.

Two strings are connected by a link of the Young graph if they differ by a simple transposition of two consecutive α, β, and new links must be added to obtain the KL-graph according to the rules given in section 2.0.4. For example, on figure 5, the KL-link connecting $\alpha\alpha\alpha\beta\beta\beta$ to $\alpha\beta\alpha\beta\alpha\beta$ is not a link of the Young graph .

By transposing the relations (5) we obtain the action of $\bar{e}_i - \tau$ on the basis states:

$$\begin{aligned} \psi_\pi(\bar{e}_i - \tau) &= \qquad -\tau\psi_\pi \quad \text{if } \pi_i\pi_{i+1} \neq \alpha\beta, \\ &= \sum_{\nu_i\nu_{i+1}\neq\alpha\beta} \mu(\nu,\pi)\psi_\nu, \text{ if } \pi_i\pi_{i+1} = \alpha\beta, \end{aligned} \tag{45}$$

which give a convenient way to generate the basis.[3]

In the case where the number of β is equal to the number of α, the K-L involution $\mathcal{T}'$ can be identified with the reflection $\mathcal{T}$ of the section 2.0.6.

3.5.2 Punctured disc basis

Our aim here is to construct the basis of polynomials dual to the punctured disc patterns of 2.0.3 using the flux representation considered in 3.4.1.

The representation cannot be generated using the Hecke algebra only and one must add the generator σ. We must enlarge the involution to include σ and we require that $\mathcal{T}'(\sigma) = \sigma$.

As a generator of the representation, we take the lowest patterns $\omega = \beta \dots \alpha$ with the β's preceding the α's. This pattern is not in the image of any e_i, and therefore, the polynomial ψ_ω must be annihilated by the $\bar{e}_i$ acting from the right. The minimal degree candidate is given by:

$$\psi_\omega(z_i) = \Delta_t(z_1, \dots, z_N). \tag{46}$$

This polynomial has a highest degree monomial given by $z_1^{N-1} z_2^{N-2} \dots z_N^0$. Thus the partition of this representation is given by:

$$\lambda^+ = N-1, N-2, \dots, 0. \tag{47}$$

ψ_ω obeys the wheel condition $k = 2,\ r = 3$, which resumes to require that for triplet $i < j < k$ the polynomial vanishes when $z_j = tz_i$ or $z_k = tz_i$ or $z_k = tz_j$. It also vanishes when two variables are set equal to zero. It therefore belongs to flux representation $p = 1, r = 3$ of section 3.4.1.

[3] Another way to proceed is to use the factorized expression of the basis states in terms of a product of Yang-Baxter operators acting on the lowest tableau [18].

The construction of the basis proceeds as in the disc case where we use the generators $\psi_{\omega\sigma^l} = \psi_\omega \bar{\sigma}^l$ to obtain the other basis states using (45) in each sector determined by l.

To characterize the pattern representation, we need to determine the parameter τ' (10). The ratio y_+/y_- of the two possible eigenvalues of y_1, obtained from (69) and from (24) must be equal. It gives, $q^{-1}t^{-1} = u^2$ if N is even, and $q^{-1}t^{-1} = u^2 t^{\frac{1}{2}}$ if N is odd, where $q = t^{-\frac{3}{2}}$ by the condition (25). Thus,

$$\begin{aligned} u &= t^{\frac{1}{2}} \text{ if } n \text{ is even,} \\ u &= 1 \text{ if } n \text{ is odd.} \end{aligned} \tag{48}$$

Therefore, in this representation, and if we parameterize $\tau = -2\cos(\theta)$, we obtain from (11) and (13):

$$\tau' = 2\cos(\frac{\theta}{2}). \tag{49}$$

We notice that in all cases, $\sigma^N = 1$. So, in order to satisfy the second duality relation (16), the normalization factor of $\bar{\sigma}$ must be taken equal to: $c_0 = q^{\frac{|\lambda|}{n}}$ in (21).

3.5.3 Conditional expectation value and Involutions

We give two properties of these polynomials parallel to the link patterns properties discussed in sections 2.0.6 and 2.0.7:

One can realize the involutions dual to those of section (2.0.6) through the transformations:

$$\begin{aligned} \bar{\mathcal{F}}(\Psi)(z_1,\dots,z_N) &= \Psi(z_N^{-1},\dots,z_1^{-1}). \\ \bar{\mathcal{T}}(\Psi_t)(z_1,\dots,z_N) &= \Psi_{t^{-1}}(z_1^{-1},\dots,z_N^{-1}). \end{aligned} \tag{50}$$

The projection E' dual to the inclusion defined in (14) is defined in the appendix (C.2) by specializing the two first variables to take the value 1 and t^2. It sends polynomials in N variables obeying the vanishing condition into polynomials in $N-2$ variables obeying the same property. By combining it with the cyclic transformation (20), it gives a way to decompose a polynomial satisfying the wheel condition in the link pattern basis by specializing its variables to be powers of t.

4 Positivity conjectures

The positivity conjectures are motivated by the R-S conjecture [4] which states that at $\tau = 1$, the evaluations of the polynomials considered in the preceding section at $z_i = 1$ count certain classes of alternating sign matrices. We claim that the evaluation of their deformations are positive polynomials in the deformation parameter τ (are in $\mathbb{N}[\tau]$).

Similarly, we observe that the evaluation of the cylinder polynomials are in $\mathbb{N}[\tau']$, where where we set $\tau = -2\cos(\theta)$ and $\tau' = 2\cos(\theta/2)$. When $\tau = \tau' = 1$, these polynomials count certain classes of half-turn symmetric alternating sign matrices [22].

A first observation concerns the K-L basis constructed from the product of Vandermonde determinants (40). We define the evaluation of a polynomial to be:

$$\bar{\psi}_\pi \;=\; N^{-1}\psi_\pi(1,\ldots,1), \tag{51}$$

where the normalization factor $N = \bar{\psi}_1(1,\ldots,1)$ in (40).

Let us show that the evaluation of K-L basis polynomials $\bar{\psi}_\pi$ are in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$. The ψ_π are polynomials in z_i with coefficients in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$. The property is true for ψ_1, and preserved under the action of $\bar{e}_i - \tau$ (18). Thus, $\bar{\psi}_\pi \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ if $\psi(1,\ldots,1)$ is divisible by $(t-1)^{\sum_l \frac{N_l(N_l-1)}{2}}$. This results from the fact that when $t-1$ and $z_i - 1$ are $O(\epsilon)$ $\forall i$, ψ_π is $O(\epsilon^{\sum_l \frac{N_l(N_l-1)}{2}})$. Again, the property is satisfied by ψ_1 and preserved by the action of $\bar{e}_i - \tau$ (and $\bar{\sigma}$).

We have verified that they are in fact *positive* in $-t^{\pm\frac{1}{2}}$.

Let us consider more specifically the link pattern polynomials.

The disc polynomials (without line connected to the center) are obtained by considering K-L representations having a Young tableau with two columns of equal length.

In this case, $\bar{\psi}_\pi$ is invariant under the transformation $t^{\frac{1}{2}} \to t^{-\frac{1}{2}}$ and is therefore in $\mathbb{Z}[\tau]$. To show this, let us consider the reflection $\mathcal{T}$ which takes $z_i \to z_i^{-1}$ and $t^{\frac{1}{2}} \to t^{-\frac{1}{2}}$ and multiplies the result by a factor $(z_1 \ldots z_N)^{(\frac{N}{2}-1)}$. If a polynomial ψ invariant under a $\mathcal{T}$ reflection, then $\bar{\psi}$ is invariant under the transformation $t^{\frac{1}{2}} \to t^{-\frac{1}{2}}$. $\mathcal{T}$ leaves ψ_1 invariant and commutes with $\bar{e}_i - \tau$ (and $t^{-\frac{3}{2}(\frac{N}{2}-1)}\bar{\sigma}$). Therefore, the ψ_π are left invariant under $\mathcal{T}$ and the property $\bar{\psi}_\pi \in \mathbb{Z}[\tau]$ follows.

We conjecture that the evaluation of the disc polynomials are positive in τ.

In table 1, we evaluate explicitly the polynomials $\bar{\psi}_\pi$ up to $N = 8$, and we observe that $\bar{\psi}_\pi \in \mathbb{N}[\tau]$.

In table 2, we evaluate the polynomials $\bar{\psi}_\pi$ with $k = 2, r = 3$, and $N = 2, 4$, obtained from the polynomial $\psi_{2,3}^{(n)}$ (36). The normalization factor in (51) is taken to be $(-)^{\frac{N}{2}}(t^{\frac{1}{4}} -$

12	1
13	τ

123	1
124	2τ
134	τ^2
125	τ^2
135	$\tau(\tau^2+1)$

1234	1
1235	3τ
1245	$3\tau^2$
1236	$3\tau^2$
1345	τ^3
1237	τ^3
1246	$\tau(5\tau^2+2)$
1256	τ^4
1346	$\tau^2(2\tau^2+1)$
1247	$\tau^2(2\tau^2+1)$
1356	$\tau^3(\tau^2+2)$
1257	$\tau^3(\tau^2+2)$
1347	$\tau(\tau^4+\tau^2+1)$
1357	$\tau^2(\tau^4+3\tau^2+3)$

Table 1: Evaluation of the disc polynomials for $k=2,\ r=2$ and $N=4,6,8$. We index the polynomials $\bar{\psi}_\pi$ by the position of the α's in the notation of 2.0.3. The evaluation is left unchanged under the flip isomorphism defined in 2.0.6 and 3.5.3: $\bar{\psi}_\pi = \bar{\psi}_{\mathcal{F}(\pi)}$.

$t^{-\frac{1}{4}})^{N(\frac{N}{2}-1)}(t^{\frac{1}{4}}+t^{-\frac{1}{4}})^{\frac{N}{2}(\frac{N}{2}-1)}$. This time, we observe that the evaluation polynomials are positive in the variable $\tau'^2 = 2-\tau$.

12	$1+\tau'^2$
13	2

123	$\tau'^6+2\tau'^4+3\tau'^2+1$
124	$2\tau'^4+8\tau'^2+4$
134	$3\tau'^2+4$
125	$3\tau'^2+4$
135	$\tau'^4+3\tau'^2+10$

Table 2: Evaluation of the disc polynomials for $k=2,\ r=3$ and $N=4,6$.

In the punctured disc case, we have considered the simplest cases where the representation is generated from the Vandermonde determinant (46) (which corresponds to the case $k=2, r=3, p=1$, in the notations of section 3.5). The normalization factor in (51) is taken to be $N = (-)^{\frac{N}{2}}(t^{\frac{1}{4}}+t^{-\frac{1}{4}})^{\frac{N}{2}(\frac{N}{2}-1)}(t^{\frac{1}{4}}-t^{-\frac{1}{4}})^{\frac{N(N-1)}{2}}$ if N is even, and $N = (t^{\frac{1}{4}}+t^{-\frac{1}{4}})^{\frac{(N-1)^2}{4}}(t^{\frac{1}{4}}-t^{-\frac{1}{4}})^{\frac{N(N-1)}{2}}$ if N is odd. One can show the same integrability conditions as in the disc case with $t^{\frac{1}{2}} \to t^{\frac{1}{4}}$. The explicit evaluations of table 3 lead us to conjecture that the evaluations are positive in τ'.

1	1
2	τ'

12	1
13	1
23	τ'^2

12	1
13	τ'^2+2
23	τ'
14	τ'
24	$\tau'(\tau'^2+2)$
34	τ'^4

125	1
123	1
145	τ'^2
234	τ'^2
135	τ'^2+3
124	τ'^2+3
245	$\tau'^2(\tau'^2+3)$
235	$\tau'^2(\tau'^2+3)$
134	$\tau'^4+\tau'^2+2$
345	τ'^6

123	1
124	τ'^2+4
125	$(\tau'^2+2)^2$
134	$(\tau'^2+2)^2$
126	τ'
234	τ'
135	$\tau'^6+3\tau'^4+11\tau'^2+10$
136	$\tau'(\tau'^2+4)$
235	$\tau'(\tau'^2+4)$
145	$\tau'(\tau'^6+\tau'^4+\tau'^2+2)$
146	$\tau'(\tau'^4+3\tau'^2+5)$
245	$\tau'(\tau'^4+3\tau'^2+5)$
156	τ'^4
345	τ'^4
236	$\tau'(\tau'^4+5\tau'^2+3)$
246	$\tau'(\tau'^6+3\tau'^4+11\tau'^2+10)$
256	$\tau'^4(\tau'^2+4)$
346	$\tau'^4(\tau'^2+4)$
356	$\tau'^4(\tau'^4+2\tau'^2+6)$
456	τ'^9

Table 3: Evaluation of the punctured-disc polynomials for $k=2,\ r=3,\ p=1$ and $N=2,3,4,5,6$. When N is odd, the natural variable is τ'^2.

5 Transfer matrix

5.1 Ψ as an eigenvector of the transfer matrix

In the disc case, at $t^3 = q = 1$, the vector Ψ (15) is an eigenstate of a commuting family transfer matrix $T(z_0) = T(z_0|z_i)$ [5]. The matrix elements of $T(z_0)$ are polynomials in the variables z_0, z_i called the spectral parameters. We give a simple proof that in the limiting case where cyclicity is recovered: $z_{i+N} = z_i$, a vector $\Psi(z_i)$ obeying the duality relations (16) can be obtained as the stationary state of a transfer matrix.

In this case, by specializing the spectral parameters of the transfer matrix, one obtains a Hamiltonian, $H = \sum_1^N e_i$, with positive integer matrix elements having $\Psi(1, \ldots, 1)$ as a stationary state (Notice that the cyclic operator σ must be defined in order to construct e_N). Thus, by the Perron-Frobenius theorem, the components of $\Psi(1, \ldots, 1)$ can be normalized to be positive integers. These integers are conjectured to count ASM [1]. At this value, the sum of the components of $\Psi(z_i)$ is a symmetric polynomial in the z_i which can be evaluated explicitly [5]. We give a derivation of this property relying on the fact that the T.L. algebra becomes non-semisimple and can be realized trivially, ($e_i = 1$), at the corresponding values of t.

Let us define the following permutation operators acting on the spins at positions i and j:

$$Y_{ij}(\frac{z_i}{z_j}) = \frac{z_j T_{ij} - z_i T_{ij}^{-1}}{z_i t^{\frac{1}{2}} - z_j t^{-\frac{1}{2}}}. \tag{52}$$

The vector Ψ can also be characterized by the conditions equivalent to (16)[7]:

$$\begin{aligned} Y_i(\frac{z_i}{z_{i+1}})\Psi(\ldots, z_i, z_{i+1} \ldots) &= \Psi(\ldots, z_{i+1}, z_i \ldots) \\ \sigma\Psi(z_1, \ldots, z_{N-1}, z_N) &= \Psi(z_2, \ldots, z_N, q^{-1} z_1). \end{aligned} \tag{53}$$

The first relation (16) can be straightforwardly verified by substituting (52) into (53). Note that he normalization of Y_{ij} has been chosen so that $Y_{ii+1} e_i = e_i$, and therefore $\bar{e}_i$ (18) projects onto a polynomial symmetric under the exchange of z_i, z_{i+1}.

It is convenient to introduce the permutation P_{ij} which act by permuting the indices i and j. One then defines the operators $X_{ij} = P_{ij} Y_{ij}(\frac{z_i}{z_j})$ obeying the Yang-Baxter equation (equivalent to the fact that the Y_{ij} are permutation operators):

$$\begin{aligned} X_{ij} X_{ji} &= 1, \\ X_{ij} X_{kl} &= X_{kl} X_{ij}, \quad \text{if } i, j \neq k, l, \end{aligned}$$

$$X_{ij}X_{ik}X_{jk} \;\; = \;\; X_{jk}X_{ik}X_{ij}, \tag{54}$$

Using the Yang Baxter operator, we can define the commuting operators $\tilde{y}_i$ by:

$$\tilde{y}_i \;\; = \;\; X_{ii-1}\dots X_{i1}\hat{q}_i\dots X_{ii+1}, \tag{55}$$

with $\hat{q}_i$ acting from the right shifts the variable z_i by a factor q (100). By commuting $\hat{q}_i$ to the left, we replace the operators X_{ij} located to its left by X_{ij+n}. When we act from the left with $\tilde{y}_i$ on Ψ, using repeatedly the first equation (53), one sees that the product of X_{ik} substitutes the variables z_i to z_{i+n} in Ψ. Conversely, the operator $\hat{q}_i$ replaces z_i with z_{i+n}. So Ψ is an eigenvector with eigenvalue 1 of $\tilde{y}_i$:

$$\tilde{y}_i\Psi = \Psi. \tag{56}$$

In what follows, we restrict to the case where $\hat{q}_i = 1$, and we identify the transfer matrix as the generating function of the $\tilde{y}_i$. The Transfer matrix has the expression:

$$T(z_0|z_i) = \operatorname*{tr}_0\{X_{01}(\frac{z_0}{z_1})X_{02}(\frac{z_0}{z_2})\dots X_{0n}(\frac{z_0}{z_N})\}, \tag{57}$$

where the partial trace (defined in section 2.0.1) is on the label 0.

It follows from (54) that two matrices with different spectral parameters z_0 and w_0 and all the other spectral parameters equal commute with each other [42][41]:

$$[T(z_0|z_i), T(w_0|z_i)] = 0. \tag{58}$$

When the shift operator $\hat{q}_i = 1$ which occurs when $q = 1$, it is straightforward to verify that $\tilde{y}_i$ is obtained by substituting z_i to z_0 in the expression of $T(z_0) = T(z_0|z_j)$ [41]:

$$\tilde{y}_i = T(z_i). \tag{59}$$

In the pattern representation, and in the case $\tau = \tau' = u = 1$, the T.L. matrices e_i (5) transform a pattern into a single pattern with a coefficient equal to one. It follows from this property, that the line-vector having all its entries equal to one is a left eigenvector of $T(z_0)$ with the eigenvalue 1. As a result, $T(z_0)$ has a right eigenvector with the eigenvalue 1 which we determine to be Ψ.

From (59), $(T(z_0) - 1)\Psi$ is a rational fraction, with a numerator of degree N in z_0. It vanishes when $z_0 = z_i$. One also has $T(0) = T(\infty) = 1$, and thus $(T(z_0) - 1)\Psi$ also vanishes when $z_0 = 0, \infty$. It is therefore equal to zero and Ψ is an eigenvector of $T(z_0)$ with the eigenvalue 1.

5.1.1 Sum of the components

At the cyclic point $q = 1$, it follows from the explicit expression (5) of the matrices e_i, that the vector χ with all its entries equal to one is a left eigenvector of the e_i's with the eigenvalue one. Thus, the scalar product $\chi.\Psi$, equal to the sum of the components of Ψ, is by the duality relation (16) a symmetric polynomial. We obtain its expression here (see also [27]).

Up to the normalization factor, this sum is determined to be the lowest degree *symmetrical* polynomial obeying the wheel condition.

For the minimal degree disc representation generated by (35) with $k = 2$, the polynomial vanishes when three ordered variables are specialized by the $r = 2$ wheel: $z_i = z, z_j = tz, z_k = t^2 z$ for $i < j < k$. It has the degree:

$$\lambda = (N-1, N-1, N-2, N-2, \ldots, 0, 0), \tag{60}$$

if the number of variables is even ($N_1 = N_2 = N$), and the first $N-1$ is erased if the number of variables is odd ($N_1 = N,\ N_2 = N-1$). This determines it to be equal to the Schur function s_λ [7].

An analogous discussion can be made in the punctured disk case. Z_N can be determined by using a recursion argument on N as follows. Let us normalize $Z_N(z_i)$ so that its highest degree monomial $Z_N(z_i)$ is equal to $z_1^{N-1} z_2^{N-2} \ldots z_N^0$. Using the projection (107), one has:

$$E'(Z_N)(z_3, \ldots) = Z_{N-2}(z_3, \ldots). \tag{61}$$

By recursion, this condition determines $Z_N(z_i)$ completely to be the product of two Schur functions:

$$Z_N(z_i) = S_{\lambda_1,N} S_{\lambda_2,N}, \tag{62}$$

where $\lambda_1 = (0, 0, 1, 1, 2, 2 \ldots)$ and $\lambda_2 = (0, 1, 1, 2, 2 \ldots)$ and each partition has N rows. Indeed, the product $S_{\lambda_1,N} S_{\lambda_2,N}$ has the same highest degree monomial as Z_N, and each Schur function factorizes as follows when one specializes the values of the two first variables:

$$\begin{aligned} S_{\lambda_1,n}(z_1 = z, z_2 = tz, z_3, \ldots) &= \prod_{i=3}^{N} (t^2 z - z_i) S_{\lambda_1,N-2}(z_3, \ldots) \\ S_{\lambda_2,n}(z_1 = z, z_2 = tz, z_3, \ldots) &= z \prod_{i=3}^{N} (t^2 z - z_i) S_{\lambda_2,N-2}(z_3, \ldots). \end{aligned} \tag{63}$$

The product has the same highest monomial as Z_N and obeys the same recursion relation (61), it is therefore equal to Z_N.

6 Spin and SOS representations

We characterize some representation of the T.L. affine algebra $\mathcal{A}_N^T(t)$ acting on on a spin or a path basis. The representation depends on two positive integer r and s, and on a continuous parameter u. When t is generic, and for $|r-s| \leq 1$, these representations are isomorphic to the patten representations of section (2.0.3).

In the spin basis, r and s are respectively the number of $+$ and $-$ spins, u is a twist parameter, and the operators y_i are realized as triangular matrices. On the other hand, the paths form an orthogonal basis and the affine generators y_i are realized as diagonal matrices in this basis.

6.1 Paths

We describe here the action of the generators of the A.H.A. on the paths of the SOS representation. This essentially the specialization to the two column tableaux of the construction done in the appendix A.1.1. The spin basis also admits a representation described in the appendix (C.3). In the generic case, the two basis are related by a triangular transformation, and the affine generators y_i are realized as triangular matrices in the spin basis and as diagonal matrices in the path basis. There is analogous of the K-L basis in the spin case known as the canonical basis [24] which we do not discuss here.

The path basis states are directed paths π on the square lattice. The path π is a sequence of lattice points $\pi_i = (x_i^+, x_i^-),\ 0 \leq i \leq N$. It starts from the origin $(0,0)$, and moves by steps of one unit towards the north-east or the south-east:

$$(x_{i+1}^+, x_{i+1}^-) = (x_i^+ + 1, x_i^-) \text{ or } (x_i^+, x_i^- + 1), \tag{64}$$

to reach the final point $(x_N^+, x_N^-) = (N_+, N_-)$. The path can also be described by $(i, h_i) = (x_i^+ + x_i^-, x_i^+ - x_i^-)$ where h_i defines the height of the path point i. Thus the path starts from the height 0 to reach the height $N_+ - N_-$ in N_+ up and N_- down steps.

The affine generator y_i acts diagonally on a path by looking at the i^{th} step between $i-1$ and i. It is equal to $\hat{y}_+(x_{i-1}^+)$ if this step is up (towards the north-east), and to $\hat{y}_-(x_{i-1}^-)$ if it is down (towards the south-east), where $\hat{y}_+(a) = y_+ t^{-a}$ and $\hat{y}_-(a) = y_- t^{-a}$.

Let us determine the expression of the T.L. generators so that the relations (86,80) are satisfied. Since e_i commutes with y_l for $l \neq i, i+1$, we require that e_i acts locally on the piece of path between $i-1$ and $i+1$.

Therefore, the projector e_i is equal to zero if $h_{i-1} \neq h_{i+1}$, and it decomposes into block matrices, $e_i = \oplus\ \delta_{h_{i-1}-h} e^h$, where e^h is equal to zero on paths such that $h_{i-1} \neq h$ or $h_{i+1} \neq h$. It acts as a two by two matrix on a pair of paths equal everywhere

except at the three consecutive points $(i-1, i, i+1)$ where their heights take the values: $[h, h-1, h]$ and $[h, h+1, h]$.

Then, writing the last relation of (80) in terms of the T.L. generators:

$$t^{-\frac{1}{2}}y_i - t^{\frac{1}{2}}y_{i+1} = y_{i+1}e_i - e_i y_i, \tag{65}$$

and substituting the matrix of e^h into this equality, we determine its diagonal elements. Finally, by requiring that e_i is proportional to a projector and satisfies (86), we determine e^h up to a diagonal similarity transformation to be given by [30]:

$$e^h = -\frac{1}{S_h}\begin{pmatrix} S_{h-1} & S_{h+1} \\ S_{h-1} & S_{h+1} \end{pmatrix}, \tag{66}$$

where S_h is defined as:

$$S_h = y_- t^{\frac{h}{2}} - y_+ t^{-\frac{h}{2}}, \tag{67}$$

and obeys the recursion relation:

$$S_{h-1} + S_{h+1} = -\tau S_h. \tag{68}$$

With this normalization, the paths form an orthogonal basis and the square of their norm is $\prod_i S_i$.

In the appendix A.1.1, we define operators Y_l which permute the two paths $h_l = h \pm 1$ when $h_{l-1} = h_{l+1} = h$.

The two possible eigenvalues $y_\pm$ of y_1 are defined up to a common factor. It is convenient to fix this normalization by setting:

$$y_+ = ut^{\frac{N_+-1}{2}},\ y_- = u^{-1}t^{\frac{N_--1}{2}}, \tag{69}$$

so that we have:

$$y_1 y_2 \dots y_N = \sigma^N = u^{N_+-N_-}. \tag{70}$$

We can identify this representation with the spin representation of the appendix (C.3) with the same value of u by taking N_+ to be the number of $+$ spins and N_- to be the number of $-$ spins.

We also recognize the pattern representations of section (2.0.3) if we identify the values of u, and take:

$$N_+ \ = \ \frac{N}{2},\ N_- = \frac{N}{2},\ \text{if } N \ \text{ is even},$$

$$N_+ = \frac{N+1}{2},\ N_- = \frac{N-1}{2},\ \text{if } N \text{ is odd.} \tag{71}$$

The spin representation coincides with the pattern representation with $N_+ = N_- = \frac{N}{2}$ if N is even and $N_+ = \frac{N+1}{2}$, $N_- = \frac{N-1}{2}$ if N is odd.

6.1.1 Restricted paths

If parameterize u as $u = t^{\frac{k}{2}}$ and define the spin of the path to be $S^z = \frac{N_+ - N_-}{2}$, we can characterize the representation by its basis states given by paths of N steps starting from the initial height $h_0 = -k - S^z$ and reaching the final height $h_N = -k + S^z$. We denote $\rho^{(N)}_{hh'}$, the representations obtained for $h_0 = h, h_N = h'$.

The factors y_+, y_- in the definition of S_h are absorbed in the redefinition of the height so that one has:

$$S_h = t^{\frac{h}{2}} - t^{-\frac{h}{2}}, \tag{72}$$

and:

$$y_+ = t^{-\frac{h}{2}} t^{\frac{N-1}{4}},\ y_- = t^{\frac{h'}{2}} t^{\frac{N-1}{4}}. \tag{73}$$

If h_0 is integer, for S_h to be defined, the height must be restricted to be strictly positive.

We can encode a path into a two lines tableau where the numbers $i = 1, 2, \ldots, n$ are successively registered in the first or the second line according to whether the i^{th} step is towards the north-east or the south-east. The abscissa of the first line are $h_0, h_0 + 1, \ldots$ and the abscissa of the second line are $1, 2, \ldots$.

For h_0 generic, one obtains in this way the standard tableau with two lines of length N_+ and N_- described in the appendix A.1.1.

7 Coxeter-Dynkin diagram representations and action of the modular group on the trace

This section lies somewhat outside the scope of the paper. We consider more specifically the representations $\rho^N_{hh'}$ of the affine T.L. algebra $\mathcal{A}^T_N(t)$, in the root of unity case. We construct representations associated to a Coxeter-Dynkin diagram which we decompose into the $\rho^N_{hh'}$. This decomposition is independent of N. It is consistent with an action of the modular group, which leaves the Coxeter-Dynkin diagram trace invariant but acts on

$\rho^N_{hh'}$. It can be viewed as a finite size version of the modular invariant partition function of conformal the field theories (CFT).[4]

If the heights defining the paths of section 6.1 are integer, we can restrict the paths to have a strictly positive height. Similarly, when t is a root of unity, $S_p = 0$ in (73) and we can restrict the height to be strictly less than p. The paths obeying these restrictions are called restricted solid on solid (RSOS) paths [42].

In the RSOS case, the basis states of $\rho^{(n)}_{hh'}$ are the paths of length N starting from the height h and ending at the height h', such that the the heights h_i obey the constraint $1 \leq h_i \leq p-1$.

We now construct a representation of $\mathcal{A}^T_N(t)$ associated to an arbitrary finite bipartite graph $\mathcal{D}$ which we identify with its incidence matrix [30]. The Hilbert $H^N_{\mathcal{D}}$ is defined by its orthogonal basis given by the closed paths of length N: $|a_0, a_1, \ldots, a_N = a_0\rangle$, drawn on $\mathcal{D}$, such that the two vertices a_i and a_{i+1} are adjacent on the graph.

Let S_a be the components of an eigenvector of $\mathcal{D}$, and $\tau = -(t+t^{-1})$ be the corresponding eigenvalue. The T.L. generators e_l are defined similarly as in (66): e_l acts locally on the piece of path between $l-1$ and $l+1$, it is equal to zero if $a_{l-1} \neq a_{l+1}$. It decomposes into block matrices $e_l = \oplus\ e^a$ where e^a acts on the pieces of path: $|a_{l-1} = a, a_l = b, a_{l+1} = a\rangle = |b\rangle$ and is given by:

$$e^a_{bc} = -\frac{S_c}{S_a}. \tag{74}$$

The cyclic operator σ cyclically shifts the paths by one unit: $\sigma|a_i\rangle = |a_{i-1}\rangle$.

These representations are particularly interesting in the case where the incidence matrix of the diagram has a Perron-Frobenius eigenvalue less than two. The diagram is then a Coxeter-Dynkin diagram $\mathcal{D} \in A_m, D_m, E_6, E_7, E_8$, and we choose S_h to be the Perron-Frobenius eigenvector of $\mathcal{D}$. t is then a primitive root of unity, $t = e^{\frac{2i\pi}{p}}$, where p is the Coxeter-number of the diagrams given below:

$$A_m, D_m, E_6, E_7, E_8 \leftrightarrow m+1, 2(m-1), 12, 18, 30. \tag{75}$$

We can decompose the Hilbert space $H^N_{\mathcal{D}}$ into the irreducible representations $\rho^N_{hh'}$:

$$H^N_{\mathcal{D}} = \oplus \gamma^{\mathcal{D}}_{hh'} \rho^N_{hh'}, \tag{76}$$

where $\gamma^{\mathcal{D}}_{hh'} \in \mathbb{N}$ are the multiplicities.

[4] A geometrical interpretation of the splitting into two independent numbers h, h' can be given from the three dimensional Topological Quantum Field Theory (or annular tensor category) point of view [40].

By adapting the arguments of [31, 32], we compute the multiplicities in the appendix E and we show that they are independent of N when it is large enough. The coefficients of the decomposition coincide with the coefficients of the character decomposition of parafermionic C.F.T. unitary models [33][34]. More precisely, one can define an action of the modular group which leaves the trace of the $\mathcal{D}$ representations invariant and transforms the trace of $\rho_{hh'}$ as the tensor product of the two characters χ_h and $\chi_{h'}$ of the Virasoro algebra entering the decomposition of the partition function.

8 Conclusion

We have deformed the stationary state of a $O(1)$ model defined on the disc and the cylinder. This has led us to study polynomial representations of the affine Hecke algebra depending on two complex parameters t and q related by the relation $t^{k+1}q^{r-1} = 1$. These polynomials obey some vanishing conditions and interpolate between the stationary state of a stochastic transfer matrix at $t^{k+1} = 1$ and a Q.H.E. wave function at $t = 1$. In the cases presented here ($k = 2$), the transfer matrix is a that of a $O(n)$ model with $n = 1$. Another family not considered here interpolates between the stationary state of a $O(1)$ model related to the Birman-Wenzl-Murakami algebra and the Pfaffian state of the Q.H.E. [45].

One can distinguish a basis labelled by link patterns which in the disc case coincides with the Kazhdan-Lusztig basis. We conjecture that the specialization of the basis polynomials at $z_i = 1$ are *positive* in the loop fugacity. Moreover, when the loop fugacity is equal to one, they count certain classes of alternating sign matrices.

It is intriguing that the spectral parameters of a transfer matrix can also be viewed as the coordinates of particles moving in the plane, so that the braiding properties of the polynomials are analogous to the fractional statistics properties of the Q.H.E. wave functions [48]. The vanishing conditions obeyed by the polynomials are the q-deformed vanishing conditions of the Q.H.E. wave functions [49], and we suspect that their positivity properties are in some way related to the incompressible (minimal area for the quantum state which translate into minimal degree condition) properties of the corresponding Q.H.E. wave functions.

Finally, in another direction, we have established that the unitary representations of the A.H.A. at roots of unity obey modular properties similar to those of conformal field theories.

8.1 Acknowledgments

We thank P. Di Francesco, T. Miwa, V. Jones, B. Nienhuis, A.V. Razumov, Y.G. Stroganov, T. Suzuki, K. Walker, P. Zinn-Justin and J-B. Zuber for interesting discussions and comments. We thank A. Lascoux for explanations which helped us to considerably improve the revised version of this paper.

This work is partially supported by Grant-in-Aid for JSPS Fellows No. 17-2106 and by the ANR program "GIMP", ANR-05-BLAN-0029-01.

8.2 Note added in proof

Since we proposed the positivity conjectures, some progress has been made confirming their validity. P. Di Francesco has conjectured that the sum of the evaluations at $z_i = 1$ of the disc polynomials (table 1) are related to a q-enumeration of Totally Symmetric Self-Complementary Plane Partitions [51]. In collaboration with A. Lascoux, one of us has established a similar relation for the highest degree coefficients in τ of these evaluations [52].

A Affine algebras

In this appendix, we give the defining relations of the double A.H.A.

The Hecke algebra depending on the parameter t, $\mathcal{A}_N(t)$ (or $\mathcal{A}_N$ when there is no possible confusion), is generated by the generators $T_1, T_2, \ldots, T_{N-1}$,[5] obeying the braid relations:

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \\ T_iT_j &= T_jT_i \text{ if } |i-j| > 1. \end{aligned} \tag{77}$$

and the quadratic relation:

$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0, \tag{78}$$

for $1 \leq i \leq n-1$.

If we define the generators $e_i = T_i - t^{\frac{1}{2}}$, the e_i are projectors obeying the Hecke relations:

$$e_i^2 = \tau e_i,$$

[5] Sometimes, we use the notation T_{ii+1} for T_i.

$$\begin{aligned} e_i e_j &= e_j e_i \ \text{ if } |i-j|>1 \\ e_i e_{i+1} e_i - e_i &= e_{i+1} e_i e_{i+1} - e_{i+1}, \end{aligned} \tag{79}$$

where $\tau = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}$.

The A.H.A. [23], is an extension of the Hecke algebra (78) by the generators $y_i,\ 1 \leq i \leq n$ obeying the following relations:

$$\begin{aligned} y_i y_j &= y_j y_i \\ T_i y_j &= y_j T_i \quad \text{if } j \neq i, i+1 \\ T_i y_{i+1} &= y_i T_i^{-1} \quad \text{if } i \leq N-1. \end{aligned} \tag{80}$$

The double A.H.A. [9, 8], is the extension of the A.H.A. obtained by adjoining to it operators denoted z_i. The z_i obey the same commutation relations (80) as the affine generators y_i with the generators T_k. It depends on an additional parameter q.

$$\begin{aligned} z_i z_j &= z_j z_i \\ T_i z_j &= z_j T_i \quad \text{if } j \neq i, i+1 \\ T_i z_{i+1} &= z_i T_i^{-1} \quad \text{if } i \leq N-1. \end{aligned} \tag{81}$$

The relations obeyed by the z_i and the y_i due to Cherednik [8] are given by:

$$\begin{aligned} y_1 z_2 y_1^{-1} z_2^{-1} &= T_1^2 \\ y_i(\prod_{j=1}^{N} z_j) &= q(\prod_{j=1}^{N} z_j) y_i, \\ z_i(\prod_{j=1}^{N} y_j) &= q^{-1}(\prod_{j=1}^{N} y_j) z_i. \end{aligned} \tag{82}$$

A more elementary presentation [9] is in term of the T_i, z_i and of a cyclic generator σ defined as:

$$\sigma = T_{N-1}^{-1} ... T_1^{-1} y_1. \tag{83}$$

We extend the definition of the variables $z_i,\ i \in \mathbb{Z}$, by cyclicity:

$$z_{i+N} = q^{-1} z_i. \tag{84}$$

Similarly, one can define a braid generator T_N by $T_N = \sigma T_1 \sigma^{-1}$. Using the braid relations again, one gets $\sigma T_N = T_{N-1}\sigma$, and one can extend the definition of T_i to $i \in \mathbb{Z}$: $T_{i+N} = T_i$.

The defining relations of σ are then:

$$\begin{aligned} \sigma T_i &= T_{i-1}\sigma, \\ \sigma z_i &= z_{i-1}\sigma. \end{aligned} \tag{85}$$

The double affine Hecke algebra is generated by the generators T_i, z_i and σ (85). In the appendix C, we reconstruct the generators y_i from this presentation

In this paper, we are often concerned with the T.L. quotient $\mathcal{A}_N^T(t)$ of the A.H.A. generated by e_i, y_i where we constrain the generators e_i to obey the restrictions:

$$e_i e_{i\pm 1} e_i \;=\; e_i. \tag{86}$$

A.1 Intertwining operators

A.1.1 Tableaux representations of the A.H.A.

Following [36], one can define operators Y_l which intertwine the affine generators:

$$\begin{aligned} y_j Y_l &= Y_l y_j \text{ if } j \neq l, l+1 \\ y_l Y_l &= Y_l y_{l+1} \\ y_{l+1} Y_l &= Y_l y_l. \end{aligned} \tag{87}$$

The relations (87) are satisfied by

$$\begin{aligned} Y_l' &= y_l T_l^{-1} - y_{l+1} T_l, \\ &= (t^{\frac{1}{2}} y_l - t^{-\frac{1}{2}} y_{l+1}) + (y_l - y_{l+1})(\tau - e_l). \end{aligned} \tag{88}$$

Thus, Y_l' intertwines the eigenstates of y_l and y_{l+1}. The square of Y_l' acts diagonally on such states:

$$Y_l'^2 = (t^{\frac{1}{2}} y_l - t^{-\frac{1}{2}} y_{l+1})(t^{\frac{1}{2}} y_{l+1} - t^{-\frac{1}{2}} y_l), \tag{89}$$

and is null on the states with $y_{l+1} = t^{\pm 1} y_l$.

We can use this property to construct a representation of the Hecke algebra on standard tableaux. A standard tableau is a right eigenvector of the y_i's and the eigenvalue of y_i depends on the box of the diagram where the number i is located. Each box is assigned an eigenvalue so that a vertical move of one unit north or a horizontal move of one unit east multiplies the eigenvalue by a factor t.

Thus, the two eigenvalues of y_l and y_{l+1} differ by a factor of t when the numbers l and $l+1$ belong to adjacent boxes of the the same column with $y_l = ty_{l+1}$, or of the same line with $y_{l+1} = ty_l$. The tableau is annihilated by Y'_l in both cases. It is annihilated by e_l in the column case and by $e_l - \tau$ in the line case.

If l and $l+1$ do not belong to adjacent boxes, Y'_l exchanges the position of l and $l+1$. We can multiply Y'_l by a normalization factor equal to $(t^{\frac{1}{2}}y_l - t^{-\frac{1}{2}}y_{l+1})^{-1}$ so that the square of the resulting operator, Y_l, is equal to one:

$$Y_l \;=\; 1 + \frac{y_l - y_{l+1}}{t^{\frac{1}{2}}y_l - t^{-\frac{1}{2}}y_{l+1}}(\tau - e_l), . \tag{90}$$

If we adopt a normalization so that Y_l permutes the two tableaux, we recover the representation [43] of the Hecke algebra on tableaux. Consider two tableaux which are exchanged by Y_l. We denote $\hat{y}$ and $\hat{y}'$ the eigenvalue of y_l and y_{l+1} on the first tableau (and of y_{l+1} and y_l on the second tableau). The expression of T_l in the basis made by these two tableaux is given by:

$$T_l = \frac{1}{\hat{y} - \hat{y}'}\begin{pmatrix} (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\hat{y} & -t^{\frac{1}{2}}\hat{y} + t^{-\frac{1}{2}}\hat{y}' \\ -t^{-\frac{1}{2}}\hat{y} + t^{\frac{1}{2}}\hat{y}' & (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\hat{y}' \end{pmatrix}. \tag{91}$$

In the path basis of section 6.1, the paths are annihilated by Y_l if $h_{l+1} \neq h_{l-1}$. Y_l acts locally on the piece of path between $l-1$ and $l+1$ as follows: it is equal to zero if $h_{l-1} \neq h_{l+1}$, and it decomposes into block matrices $Y_l = \oplus\, Y_l^h$ where Y_l^h acts on paths with $h_{l-1} = h_{l+1} = h$ by exchanging the two paths with the intermediate heights $h_l = h-1$ and $h_l = h+1$. All the paths are obtained from one of them under the repeated action of the generators Y_l.

In the cases where h_l are integers, the paths must be restricted to have $h_l > 0$. Furthermore, when $t^p = 1$, the paths must be restricted to have $h_l < p$.

A.1.2 Shift operators

Following [36], we give the expression of an operator which changes the degree of a polynomial by keeping it an eigenstate of the operators y_i.

As in (84) for the coordinates z_i, it is convenient to extend the definition of the affine generators y_i to $i \in \mathbb{Z}$ by periodicity:

$$y_{i+N} = q^{-1} y_i. \tag{92}$$

Using the affine Hecke relations (81), one can construct a shift operator $\bar{A}$ which shifts by one unit the affine generators:

$$\begin{array}{rcl} \bar{A}\bar{y}_i & = & \bar{y}_{i+1}\bar{A} \\ \bar{A}\bar{g}_i & = & \bar{g}_{i+1}\bar{A} \end{array} \tag{93}$$

It raises the degree of the polynomial by one and its expression is given by

$$\bar{A} \;=\; z_1 \bar{\sigma}^{-1}. \tag{94}$$

B Explicit matrices in the cases n=2,3

In the basis $\alpha\beta, \beta\alpha$, the action of $\mathcal{A}_2^T$ is generated by the matrices:

$$e_1 = \begin{pmatrix} \tau & \tau' \\ 0 & 0 \end{pmatrix}, \; \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{95}$$

The T.L. algebra relation (86) is modified into: $e_1 e_2 e_1 = \tau'^2 e_1$ because the left hand side creates two lines around the torus in this case.

In the basis $\alpha\alpha\beta$, $\alpha\beta\alpha$, $\beta\alpha\alpha$ the generators of $\mathcal{A}_3^T$ are given by:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \tau & u \\ 0 & 0 & 0 \end{pmatrix}, \; \sigma = \begin{pmatrix} 0 & 0 & 1 \\ u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{96}$$

The matrix y_1 is given by:

$$y_1 = \begin{pmatrix} u & u^{-1} & -t^{-\frac{1}{2}} \\ 0 & u^{-1}t^{-\frac{1}{2}} + u & -t^{-1} \\ 0 & t^{\frac{1}{2}} & 0 \end{pmatrix}. \tag{97}$$

C Representation of the $\bar{y}_j$ in the polynomial and the spin cases.

C.1 polynomial representation

We repeat here the diagonalization of the operators $\bar{y}_j$ done in [9]. The method follows the one initiated by Sutherland [50] in the Calogero-Sutherland model context.

For a certain ordering of the monomial basis, we obtain an expression of y_i in a triangular form by decomposing it into a product of triangular matrices x_{ij} and a diagonal operator $\hat{q}_i$.

The expression of $\bar{y}_i$ which follows from (83) and (80) is given by:

$$\bar{y}_i \;=\; \bar{T}_i\bar{T}_{i+1}\dots\bar{T}_{N-1}\bar{\sigma}\bar{T}_1^{-1}\dots\bar{T}_{i-1}^{-1}. \tag{98}$$

It is convenient to reexpress y_i as a product of triangular operators. for this, we decompose the cyclic operator sigma as a product of elementary permutation s_i which permute the variables z_i and z_{i+1} and an operator $\hat{q}_1$ which replace the variable z_1 with $q^{-1}z_1$:

$$\bar{\sigma} = c_0 s_{N-1}\dots s_2 s_1\hat{q}_1. \tag{99}$$

with $\hat{q}_i$ given by:

$$P(z_1,\dots,z_i\dots,z_N)\hat{q}_i \;=\; P(z_1\dots,q^{-1}z_i,\dots,z_N). \tag{100}$$

We define $x_{ij} = \bar{T}_{ij}s_{ij}$ where s_{ij} is the permutation operator which permutes the variables z_i and z_j. $\bar{T}_{ij}$ is defined in the same way as $\bar{T}_i = \bar{T}_{ii+1} = t^{\frac{1}{2}} + \bar{e}_i$ where we replace the variable z_{i+1} with the variable z_j and the permutation s_i with permutation s_{ij} in the expression (18) of $\bar{e}_i$. The operator x_{ij} takes the form for $i < j$:

$$x_{ij} = -t^{-\frac{1}{2}} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(1 - s_{ij})\frac{z_j}{z_i - z_j}, \tag{101}$$

After permuting the s_i through, the expression of $\bar{y}_i$ becomes:

$$\bar{y}_i \;=\; c_0 x_{ii+1}x_{ii+2}\dots x_{iN}\hat{q}_i x_{1i}^{-1}\dots x_{i-1i}^{-1} \tag{102}$$

The operator x_{ij} commutes with z_iz_j and acts as a triangular matrix on the monomials $z_i^m,\ z_j^m$:

$$z_i^m x_{ij} \;=\; -t^{-\frac{1}{2}}z_i^m + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(z_i^{m-1}z_j + z_i^{m-2}z_j^2 + \dots + z_j^m) \ \text{ if } m \geq 0,$$

$$z_j^m x_{ij} \;=\; -t^{\frac{1}{2}} z_j^m - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(z_i^{m-1} z_j + z_i^{m-2} z_j^2 + \ldots + z_i z_j^{m-1}) \text{ if } m > 0. \quad (103)$$

Thus, the operators $\bar{y}_i$ are also realized as triangular operators in the monomial basis.

We order on the monomials by saying that $z^\lambda = z_1^{\lambda_1} \ldots z_1^{\lambda_N}$ is larger than $z^\mu = z_1^{\mu_1} \ldots z_N^{\mu_N}$ if either μ is obtained from λ by a sequence of squeezing operations $\{\lambda'_i, \lambda'_j\} \to \{\lambda'_i - 1, \lambda'_j + 1\}$ with $\lambda'_i > \lambda'_j + 1$, or μ is a permutation of λ and can be obtained from λ by a sequence of permutations $(\lambda'_i, \lambda'_{i+1}) \to (\lambda'_{i+1}, \lambda'_i)$ with $\lambda'_i > \lambda'_{i+1}$.

It follows from the expressions (103) that the action of y_j on a monomial produces only monomials which are smaller with respect to this order, and their eigenvalues are given by the diagonal elements in the monomial basis.

A common eigenstate of the operators $\bar{y}_i$ is characterized by its highest degree monomial z^{λ_π} where π is the shortest permutation such that $(\lambda_\pi)_i = \lambda^+_{\pi_i}$ and λ^+ is a partition $(\lambda^+_1 \geq \lambda^+_2 \ldots \geq \lambda^+_N)$.

The action of $\bar{y}_j$ on a monomial z^{λ_π} is given by

$$z^{\lambda_\pi} \bar{y}_j = c_0 (-t^{-\frac{1}{2}})^{N-1} q^{-\lambda^+_{\pi_j}} t^{\pi_j - 1} z^{\lambda_\pi} + \text{lower terms}, \quad (104)$$

from which the eigenvalues of $\bar{y}_j$ follow.

The global normalization of $\bar{y}_i$ is such that:

$$\bar{y}_1 \bar{y}_2 \ldots \bar{y}_N = \bar{\sigma}^N = c_0^N \hat{q}_1 \ldots \hat{q}_N = c_0^N q^{-|\lambda|}. \quad (105)$$

C.2 Conditional expectation value in the T.L. cases

When the A.H.A. reduces to a T.L. algebra $\mathcal{A}_N^T$, in the case of minimal degree polynomials with $k = 2$, we can define a projection E' from the polynomials in N variables to the polynomials in $N - 2$ variables dual to the inclusion defined in (14).

For any polynomial F, E' satisfies the conditions:

$$\begin{array}{ll} a) & E'(Fe_1) = \tau E'(F) \\ b) & E'(Fx) = E'(F)x \;\forall x \in \mathcal{A}^T_{N-2} \\ c) & E'(Fe_1) = 0 \Rightarrow Fe_1 = 0, \end{array} \quad (106)$$

and can be realized as:

$$E'(F)(z_3, z_4, \ldots) = \phi(z, z_3, z_4, \ldots)^{-1} F(z_1 = z, z_2 = tz, z_3, z_4, \ldots), \quad (107)$$

where $\phi(z, z_i)$ is a polynomial which removes the z dependence of F in the right hand side of (107) equal to:

$$\phi(z, z_i) \;=\; z^p \prod_{i=3}^{N} \prod_{b=0}^{r-2} (t^2 q^b z - z_i). \quad (108)$$

C.3 Spin representation

The spin 1/2 representation of $\mathcal{A}_N$ can be obtained from the representation on polynomials with a degree less or equal to 1 in each variable. The monomials z^λ are the spin basis elements: $\lambda_i = 1$ if the spin i is plus and 0 if it is minus.

Let us describe this representation explicitly. The Hilbert space is $(\mathbb{C}^2)^N$ with a basis given by sequences of spins $|\pm\pm\ldots\pm\rangle$. The matrix e_i acts in $\mathbb{C}^2_i \otimes \mathbb{C}^2_{i+1}$ and has the following expression in the basis $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$:

$$e_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -t^{-\frac{1}{2}} & 1 & 0 \\ 0 & 1 & -t^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{109}$$

The permutation operators P_{ij} permute the spins at positions i and j. It is convenient to introduce the operators e_{ij} having the expression (109) and acting in $\mathbb{C}^2_i \otimes \mathbb{C}^2_j$. $T_{ij} = t^{\frac{1}{2}} + e_{ij}$, and the operators x_{ij} are defined as in (C.1), $x_{ij} = T_{ij}P_{ij}$. In the same basis as above, x_{ij} is realized as a triangular matrices as:

$$x_{ij} = \begin{pmatrix} t^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 1 & t^{\frac{1}{2}} - t^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{\frac{1}{2}} \end{pmatrix}. \tag{110}$$

The diagonal matrix Ω_i acts on the spin at position i. In the basis $|+\rangle, |-\rangle$, it is given by:

$$\Omega_i = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}. \tag{111}$$

The matrix σ is defined by (99) where we substitute Ω_i to $\hat{q}_i$ and P_{ii+1} to s_{ii+1}:

$$\sigma = P_{N-1N} \ldots P_{12}\Omega_1. \tag{112}$$

It implies the following identification of spins: $\mu_{i+N} = u^{-\mu_i}\mu_i$.

The Hilbert space is characterized by the total numbers $N_+,\ N_-$, of plus and minus spins respectively ($N_+ + N_- = N$). The representation also depends on the parameter u of the twist matrix Ω.

We can give an alternative definition of the spin representation with the total spin $\frac{N_+-N_-}{2}$. Let $\mathcal{A}^T_{(N_+,N_-)}$ be a subalgebra of $\mathcal{A}^T_N$ generated by $T_1,\ldots,T_{N_+-1},T_{N_++1},\ldots,T_{N-1}$, and $y_1,\ldots,y_N$. We define a one-dimensional representation $\mathbb{C}\mathbf{1}$ of $\mathcal{A}_{(N_+,N_-)}$ as follows:

$$
\begin{aligned}
T_i\mathbf{1} &= t^{\frac{1}{2}}\mathbf{1}, && (113)\\
y_i\mathbf{1} &= ut^{\frac{N_+-1}{2}}t^{i-1}\mathbf{1} \quad \text{if } 1\le i\le N_+, && (114)\\
y_i\mathbf{1} &= u^{-1}t^{\frac{N_--1}{2}}t^{i-N_+-1}\mathbf{1} \quad \text{if } N_++1\le i\le N. && (115)
\end{aligned}
$$

Then the spin representation is isomorphic to an induced module $\mathrm{Ind}^{\mathcal{A}_N}_{\mathcal{A}_{(N_+,N_-)}}\mathbb{C}\mathbf{1}$, where $|++\cdots+--\cdots-\rangle$ corresponds to $\mathbf{1}$.

We can define an ordering on the basis as follows. A state $|\mu'\rangle$ is smaller than $|\mu\rangle$, if it can be obtained from $|\mu\rangle$ through a sequence of permutations of a plus spin at position i and a minus spin at position $i+1$.

D Stability of the wheel condition under the action of the A.H.A.

We show here that the wheel conditions are preserved under the action of the A.H.A. generators.

Let us verify that the ideal of polynomials obeying the condition (27) under the restriction (26a) is preserved under the action of $\bar{e}_i$ (18).

Consider the polynomial $P' = P\bar{e}_i$ where $P(z_i)$ is a polynomial obeying any admissible wheel condition. We show that P' obeys the wheel condition specified by any admissible wheel $\{i_a\}$ and $\{b_{aa+1}\}$.

If there exists a value $\bar{a}$, such that $i = i_{\bar{a}}$, $i+1 = i_{\bar{a}+1}$, and $b_{\bar{a}\bar{a}+1} = 0$, then $\tau P - P'$ is proportional to $t^{\frac{1}{2}}z_i - t^{-\frac{1}{2}}z_{i+1}$ and obeys this wheel condition. By linearity, so does P'.

If not, the wheel deduced by permutation: $i'_a = s_i(i_a)$ and $b'_{aa+1} = b_{aa+1}$ $\forall a$ is admissible. Thus, Ps_i, and by linearity P', also obey this wheel condition.

Let us show that the conditions (25) and (26b) imply that the space of polynomials obeying the wheel conditions is preserved under the action of $\bar{\sigma}$ (20). This amounts to show that the transformation defined by $i_a \to i_a+1$ with $\{b_{aa+1}\}$ unchanged defines an admissible wheel condition. The identification $z_{N+1} = q^{-1}z_1$ (19), is used when $i_a+1 = N+1$.

If $i_a < N$ $\forall a$, it is obvious.

If $i_{\bar{a}} = N$ for a value $\bar{a} \neq k+1$, then $i_{\bar{a}+1} < N$ and $b_{\bar{a}\bar{a}+1} \geq 1$. The above transformation can be recast into the wheel condition: $i'_a = i_a + 1$ for $a \neq \bar{a}$, $i'_{\bar{a}} = 1$, and $b'_{\bar{a}-1\bar{a}} = b_{\bar{a}-1\bar{a}} + 1$, $b'_{\bar{a}\bar{a}+1} = b_{\bar{a}\bar{a}+1} - 1$, $b'_{aa+1} = b'_{aa+1}$ for a and $a+1 \neq \bar{a}$.

If $i_{k+1} = N$, this transformation can be recast into the wheel condition: $i'_a = i_{a-1}+1$ for $a \geq 2$, $i'_1 = 1$, and $b'_{aa+1} = b_{a-1a}$ for $a \geq 2$, $b_{12} = r - 2 - \sum_1^k b_{aa+1}$. To express b_{12} in this form we have used the condition (25), and the condition (26b) is necessary to have $b_{12} \geq 0$. We also have: $\sum_1^k b'_{aa+1} \leq r-2$ and the condition (26b) is satisfied by the new wheel.

E Partition functions and traces

In this appendix, we give a graphical method to evaluate the trace of operators in $\mathcal{A}_N^T(t)$ following [31]. The trace depends on the representation. We compare the traces between the different representations, and obtain the decomposition of $\rho_{\mathcal{D}}$ into the representations $\rho_{hh'}$, by showing the trace identity:

$$\mathrm{tr}_{\mathcal{D}}(x) = \sum_{hh'} \gamma^{\mathcal{D}}_{hh'} \mathrm{tr}_{\rho_{hh'}}(x), \forall x \in \mathcal{A}_N^T(t). \tag{116}$$

In this identity, the matrices are finite dimensional. We fix t to be the root of unity, $t = e^{\frac{2i\pi}{p}}$, where p is the Coxeter number of the diagram $\mathcal{D}$.

We also define an action of the modular group which leaves $\mathrm{tr}_{\mathcal{D}}$ invariant but transforms linearly the traces $\mathrm{tr}_{\rho_{hh'}}$. The coefficients $\gamma^{\mathcal{D}}_{hh'}$ therefore define a modular invariant decomposition of the trace.

To represent the trace, we use the description of linear operators by a system of lines drawn on the annulus of section 2.0.3. We close the annulus into a torus by identifying the two boundaries. The trace becomes the partition function of a loop model on the torus.

The contractible loops can be removed by giving them a weight τ. One ends up with a system of non-contractible loops not touching each other and therefore homotopic to the same cycle.

In the case of the models defined by a diagram $\mathcal{D}$ of section 7, if the number loops is $2m$, the weight is the number of paths of length $2m$ which can be drawn on the diagram $\mathcal{D}$.

In the case of the spin representation of the appendix C.3, the lines carry a spin index and are oriented accordingly. We must then sum over the possible orientations of the

loops. We cut the annulus into a rectangle as in figure 6. If the total spin across a horizontal cycle is $2S^z$, and the total spin across a vertical cycle is $2S^{z\prime}$. The weight of an oriented loop configuration is $u^{2S^{z\prime}}$.

We denote by $\binom{k}{S^z}$ the spin-representation of $\mathcal{A}_N^T(t)$ where the value of the spin is fixed to be S^z and $u = t^{\frac{k}{2}}$. If $|S^z| > \frac{N}{2}$, we regard $\binom{k}{S^z}$ as the zero representation.

We define two actions s_0 and s_1 on a 2-tuple $\binom{a}{b}$ by:

$$s_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \tag{117}$$

$$s_0 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b-p \\ a+p \end{pmatrix}. \tag{118}$$

Setting $h = -k - S^z$ and $h' = -k + S^z$, we conjecture that:

$$\mathrm{tr}_{\rho_{hh'}}(x) = \sum_{w \in W} (-1)^{l(w)} \mathrm{tr}_{w\binom{k}{S^z}}(x) \quad , \forall x \in \mathcal{A}_N^T(t) \tag{119}$$

where $W = W(A_1^{(1)})$ is the affine Weyl group of type $A_1^{(1)}$. Note that the right hand side is a finite sum.

To establish the identity (116), it is useful to introduce an intermediate representation ρ_f defined by a graph $\mathcal{D}_f$ made of $2f$ vertices connected around a circle. We require that f divides p so that $\tau = -(t+t^{-1})$ is an eigenvalue of $\mathcal{D}_f$. We obtain a representation of $\mathcal{A}_N^T(t)$ if we define the T.L. generators e_l as in (74) by taking for S_a the eigenvector of $\mathcal{D}_f$ with the eigenvalue τ. We can also view the trace of this representation as the partition function of a spin model where the total spins $2S^z$ and $2S^{z\prime}$ across the horizontal and vertical cycles considered above are constrained to be equal to f modulo p. We set $p = ff'$. By representing the constraint on $2S^{z\prime}$ as a Fourier sum, we obtain the decomposition:

$$\rho_f = 2 \oplus \begin{pmatrix} af' \\ bf \end{pmatrix} \text{ with } 0 \leq a \leq f-1, \; 0 \leq b \leq f'-1. \tag{120}$$

We can decompose the representations $\rho_{\mathcal{D}}$ in terms of the ρ_f by identifying their traces as in (116). With this interpretation we write $\rho_{\mathcal{D}} = \sum c_f \rho_f$ where f is a divisor of p. From the property of the trace, we determine the coefficients c_f by requiring that the number of closed paths of a given length on the graph $\mathcal{D}$ is the same as the sum over f of the number of closed paths of the same length on the circular diagram $\mathcal{D}_f$ multiplied by c_f. One obtains [31]:

$$2\rho_{A_n} = \rho_{n+1} - \rho_1,$$

$$
\begin{aligned}
2\rho_{D_n} &= \rho_{2(n-1)} - \rho_{n-1} + \rho_2 - \rho_1, \\
2\rho_{E_6} &= \rho_{12} - \rho_6 - \rho_4 + \rho_3 + \rho_2 - \rho_1, \\
2\rho_{E_7} &= \rho_{18} - \rho_9 - \rho_6 + \rho_3 + \rho_2 - \rho_1, \\
2\rho_{E_8} &= \rho_{30} - \rho_{15} - \rho_{10} - \rho_6 + \rho_5 + \rho_3 + \rho_2 - \rho_1.
\end{aligned} \tag{121}
$$

Finally, by combining (119, 120 and 121), we obtain the following decomposition:

$$
\begin{aligned}
\rho_{A_n} &= \oplus_{k=1}^{n} \rho_{k,k}, \\
\rho_{D_{2n}} &= \oplus_{k=1}^{n-1} \rho_{(2k-1)+(4n-2k-1),(2k-1)+(4n-2k-1)} \oplus 2\rho_{2n-1,2n-1}, \\
\rho_{D_{2n+1}} &= \oplus_{k=1}^{2n} \rho_{2k-1,2k-1} \oplus \rho_{2n,2n} \oplus_{k=1}^{n-1} \left(\rho_{2k,4n-k} \oplus \rho_{4n-k,2k}\right), \\
\rho_{E_6} &= \rho_{1+7,1+7} \oplus \rho_{4+8,4+8} \oplus \rho_{5+11,5+11}, \\
\rho_{E_7} &= \rho_{1+17,1+17} \oplus \rho_{5+13,5+13} \oplus \rho_{7+11,7+11} \oplus \rho_{9,9} \oplus \rho_{5+13,9} \oplus \rho_{9,5+13}, \\
\rho_{E_8} &= \rho_{1+11+19+29,1+11+19+29} \oplus \rho_{7+13+17+23,7+13+17+23}.
\end{aligned} \tag{122}
$$

where $\rho_{a+b,c+d} = \rho_{a,c} \oplus \rho_{a,d} \oplus \rho_{b,c} \oplus \rho_{b,d}$.

Let us obtain the transformation law of the trace of $\rho_{hh'}$ under a modular transformation. The spin across the vertical and horizontal cycles is transformed as:

$$
\begin{pmatrix} S^{z\prime} \\ S^z \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} S^{z\prime} \\ S^z \end{pmatrix}. \tag{123}
$$

From the characterization preceding (119) of $\rho_{hh'}$, it is straightforward to obtain the following transformation of the traces:

$$
\begin{aligned}
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &: \quad \mathrm{tr}_{\rho_{hh'}} \rightarrow t^{\frac{1}{4}(h'^2-h^2)} \mathrm{tr}_{\rho_{hh'}}, \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} &: \quad \mathrm{tr}_{\rho_{hh'}} \rightarrow \sum_{rr'} t^{\frac{1}{4}((r-h)^2-(r'-h')^2)} \mathrm{tr}_{\rho_{rr'}}.
\end{aligned} \tag{124}
$$

Notice that under a modular transformation, the representation $\rho_{hh'}$ behaves as a tensor product of affine characters of $A_1^{(1)}$: $\rho_{hh'} \sim \chi_l \otimes \bar{\chi}_{l'}$, where the level k is given by $p = 2(k+2)$ and the spin l is given by $h = 2l+1$.

Under a modular transformation of the torus, the partition function of the $\mathcal{D}$-models remains invariant but the partition functions of $\rho_{hh'}$ transforms linearly. Therefore, the multiplicities $\gamma_{hh'}$ in (122) are such that the direct sum is left invariant under these transformations.

References

[1] A.V. Razumov and Y.G. Stroganov, *Spin chains and combinatorics,* J.Phys. A **34**, 3185, [cond-mat/0012141].

[2] P.A. Pearce, V. Rittenberg, J. de Gier and B. Nienhuis, *Temperley-Lieb Stochastic Processes* J.Phys.A **35** L661-668 (2002) [math-phys/0209017].

[3] M.T. Batchelor, J. de Gier and B.Nienhuis, J.Phys.A **34** L265-270 (2001) [cond-mat/0101385].

[4] A.V. Razumov and Y.G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop models* Theor.Math.Phys. **138** 333-337 (2004).

[5] P. Di Francesco and P. Zinn-Justin, *Around the Razumov-Stroganov conjecture: proof of a multiparameter sum rule,* Electr.J.Combin. **12**, R6 (2005), [math-ph/0410061].

[6] M. Kasatani, *Subrepresentations in the polynomial representation of the double affine Hecke algebra of type GL_n at $t^{k+1}q^{r-1} = 1$,* Int. Math. Res. Not. **2005**, no. 28, 1717–1742 [math. QA/0501272].

[7] V. Pasquier, *Quantum incompressibility and Razumov stroganov type conjectures*, Ann. Henri Poincaré **7** 397-421 (2006) [cond-mat/0506075].

[8] I. Cherednik, *Double Affine Hecke Algebras.* Cambridge University Press (2005).

[9] D. Bernard, M. Gaudin, D. Haldane and V. Pasquier, *Yang-Baxter equation in spin chains with long range interactions*, J.Phys. A **26**, 5219-5236 (1993), [hep-th/9301084].

[10] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Symmetric polynomials vanishing on the shifted diagonal and Macdonald polynomials*, Int. Math. Res. Not. **18**, 101 (2003).

[11] I.G. Macdonald, *A new class of symmetric functions.* Actes 20^e Seminaire Lotharingien, p 131-171, Publications I.R.M.A. Strasbourg (1988), 372/S-20.

[12] I.G. Macdonald, *Symmetric Functions and Hall Polynomials.*, Oxford University Press, (1995).

[13] E. Prange and S. Girvin, *The Quantum Hall effect.* Springer-Verlag, (1987).

[14] F.D.M. Haldane and E.H. Rezayi, Phys. *Spin-singlet wave function for the half-integral quantum Hall effect,* Rev. Lett. **60**, 956, and **E60**, 1886 (1988).

[15] D. Kazhdan and G. Lusztig, *Representation of Coxeter groups and Hecke algebras,* Invent.Math. **53**, 165-184 (1979).

[16] A. Lascoux, M.-P. Schützenberger, *Polynômes de Kazhdan* & *Lusztig pour les grassmaniennes*, Asterisque **87-88**, 249-266 (1981).

[17] A. Lascoux, M.-P. Schützenberger, *Symmetry and Flag manifolds*, Invariant Theory, Springer L.N **996**, 118-144 (1983).

[18] A. Kirillov, Jr. and A. Lascoux, *Factorization of Kazhdan-Lusztig elements for Grassmanians* Adv. Stud. **28** 143-154 (2000), [math/9902072].

[19] W. Fulton *Young Tableaux,* London Mathematical Society Student Texts **35** (2003) Cambridge University Press.

[20] S. Mitra and B. Nienhuis, *Osculating Random Walks on Cylinders* [math-ph/03120336].

[21] S. Mitra and B. Nienhuis, *Exact conjectured expressions for correlations in the dense $O(1)$ loop model on cylinders* [cond-mat/0407578].

[22] A.V. Razumov and Y.G. Stroganov, *Enumeration of half-turn symmetric alternating sign matrices of odd order,* [math-phys/0504022].

[23] G. Lusztig, *Affine Hecke algebras and their graded version,* J.Amer.Math.Soc. **2**, 599-635 (1989).

[24] G. Lusztig, *Introduction to Quantum Groups,* Birkhauser, Boston, (1993).

[25] I. Cherednik, *Nonsymmetric Macdonald Polynomials,* Internat. Math. Res. Notices **10**, 483-515 (1995).

[26] P. Di Francesco and P. Zinn-Justin, *Inhomogenous model of crossing loops and multidegrees of some algebraic varieties* [math-ph/0412031].

[27] P. Di Francesco, P. Zinn-Justin and J.B. Zuber, *Sum rules for the ground states of the O(1) loop model on a cylinder and the XXZ spin chain,* [math-ph/0603009].

[28] F.M. Goodman, P. de la Harpe, V.F.R. Jones, *Coxeter Graphs and Towers of Algebras.* Springer-Verlag (1989).

[29] J.J. Graham and G.I. Lehrer, Enseign.Math. *The representation theory of affine Temperley-Lieb algebras* **44** (1998).

[30] V. Pasquier, *Two-dimentional critical systems labelled by Dynkin diagrams.* Nucl.Phys.B **FS19**, 162-172 (1987).

[31] V.Pasquier, *Lattice derivation of modular invarient partition functions on the torus* J.Phys.A **20** L1229 (1987).

[32] V. Pasquier and H. Saleur, *Common structures between finite systems and conformald field theories through quantum groups* Nucl.Phys.B **330**, 525-556 (1990).

[33] A. Cappelli, C. Itzykson and J.B. Zuber, *Modular invariant partition functions in two dimensions* Nucl.Phys.B **280**[FS18], 445-465 (1987).

[34] D.Gepner and Z.Qiu, *Modular invariant partition functions for parafermionic field theories* Nucl.Phys.B **285**[FS19], 423-453 (1987).

[35] E. Prange and S. Girvin, *The Quantum Hall effect.* Springer-Verlag, (1987).

[36] F. Knop and S. Sahi, *A recursion and a combinatorial formula for Jack polynomials,* Invent. Math. **128** no.1, 9-22 (1997) [q-alg/9610016].

[37] P. Di Francesco and P. Zinn-Justin, *The quantum Knizhnik-Zamolodchikov equation, generalized Razumov-Stroganov sum rules and extended Joseph polynomials,* J.Phys.A **38** L815-822 (2002) [math-ph/0508059].

[38] A. Ram *Affine Hecke algebras and generalized standard Young tableaux*, J. Algebra **260** (2003), no. 1, 367–415 [math.RT/0401329].

[39] T. Suzuki and M. Vazirani, *Tableaux on periodic skew diagrams and irreducible representations of the double affine Hecke algebra of type A*, Int. Math. Res. Not. **2005**, no. 27, 1621–1656 [math.QA/0406617].

[40] K. Walker, private communication.

[41] M. Gaudin, *La Fonction d'Onde de Bethe,* Masson (1981).

[42] R.J. Baxter, *Exactly solved Models in Statistical Mechanics,* Academic, London (1982).

[43] H. Wenzl, *Hecke Algebras of type A_n and subfactors,* Invent.Math. **92**, 349-383 (1988).

[44] V. Pasquier, *Scattering Matrices and Affine Hecke Algebras*, Schladming School 1995, Nucl.Phys.B (Proc.Suppl.) **45A**, 62-73(1996), [q-alg/9508002].

[45] V. Pasquier, *Incompressible representations of the Birman Wenzl algebra,* Ann. Henri Poincaré **7** 603-619 (2006) [math.QA/0507364].

[46] V. Pasquier, *A lecture on the Calogero Sutherland models*, The third Baltic Rim Student Seminar, Saclay preprint, Spht-94060 (1994), [hep-th/9405104].

[47] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, and V. Pasquier, *Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory*, Phys. Rev. Lett. **69**, 20212025 (1992).

[48] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific.

[49] G. Moore and N. Read, *Nonabelions in the fractional quantum hall effect,* , Nucl.Phys. B**360** 362-91 (1991).

[50] B. Sutherland, Phys.Rev.A **5**, 1372 (1972).

[51] P. Di Francesco, *Totally Symmetric Self-Complementary Plane Partitions and Quantum Knizhnik-Zamolodchikov equation: a conjecture.* [cond-mat/0607499].

[52] A. Lascoux and V. Pasquier, in preparation.